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STABILIZATION OF DYNAMICAL SYSTEMS UNDER PERSISTENT PERTURBATIONS†

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The classical problem of the stabilization of dynamical systems is solved by methods of the theory of optimal processes. An auxiliary optimal control problem includes a test perturbation. An optimal control of the feedback type is constructed for each generalized position, using an open-loop control which is optimal for the test perturbation. Such controls generalize classical feedback in the sense that no allowance is made for predicted perturbations.

In view of the need to allow for interference in stabilizer design, attention has recently been given to the relevant questions in stability theory and control theory. There are three main trends: the theory of stability with random perturbations [1], the theory of linear-quadratic optimization with random perturbations [2], and H_∞ control theory with deterministic perturbations [3, 4].

In this paper a new type of feedback is proposed for stabilizing dynamical systems. First, the construction is carried out for a special optimal control problem, not of the traditional linear-quadratic type, involving direct restrictions on the control. In addition, the auxiliary problem contains perturbations, thus differing from models underlying classical feedback design. Finally, the perturbations are assumed to be non-stochastic, but, being deterministic, allowance for them follows techniques of neither H^∞ control theory nor guaranteed optimization theory. The feedback used in this paper is in a sense intermediate in position between classical feedback and the feedback studied in H_∞ control theory. A special case of our stabilization method, based on classical optimal feedback design, was published previously [5].

1. STATEMENT OF THE PROBLEM

Suppose that the dynamical system to be stabilized is described, together with the control used for the purpose, by the equation

$$\dot{x} = Ax + bu, \quad (x \in R^n, \ u \in R) \tag{1.1}$$

The actual states $x^*(\tau)$ of the system at any time τ are produced by the action of a control applied up to time τ, say $u^*(t)$, $t \in T_0 = [0, \tau[$, and a perturbation $w^*(t)$, $t \in T_0$, i.e. $x^*(t)$, $t \in T_0$, is a solution of the equation

$$\dot{x} = Ax + bu^*(t) + qw^*(t), \qquad x(0) = x_0 \tag{1.2}$$

We shall assume throughout that at any time $\tau \geq 0$ the values of the state vector $x^*(\tau)$ can be measured accurately.

To set up each value $u^*(\tau)$ of the control we shall use the available information on possible realizations of the noise, according to which the perturbations affecting the system, $w^*(\tau)$, $\tau \geq 0$, are elements of a given set W. There are three types of usable information on the perturbations: (1) classical information, in which it is assumed that up to the present (time τ) the perturbation is effective, but not after τ, i.e. the control system is described up to time τ by system (1.2) but for $t \geq \tau$ by Eq. (1.1); (2) probabilistic information, in which W is a set of stochastic processes; (3) guaranteed, in which W is a set of piecewise-continuous (or measurable) functions with values in a given bounded set.

The basic idea of the approach proposed here to the stabilization of dynamical systems subject to persistent perturbations is to use test perturbations: at each given time τ, information on the controls

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activated up to time τ, say $x^*_\tau(\cdot) = (x^*(t), t \in T_0)$, and on the actually realized trajectory $x^*_\tau(\cdot) = (x^*(t), t \in T_0)$ is used to construct a test perturbation $\tilde{w}_\tau(\cdot) = (\tilde{w}_\tau(t), t \in T_\tau = [\tau, \tau + \theta[)$ and an auxiliary problem is solved; namely, to determine the optimal damping of the system by a control of minimum intensity in a finite time interval

$$\begin{aligned} &\rho \to \min, \quad \dot{x} = Ax + bu + q\tilde{w}_\tau, \quad x(\tau) = x^*(\tau) \\ &x(\tau+\theta) = 0, \quad |u(t)| \leq \rho, \quad t \in T_\tau \end{aligned} \tag{1.3}$$

where θ is a finite control level.

Let $u^0(t\,|\,x^*(\tau), \tilde{w}_\tau(\cdot)), t \in T_\tau$ be an optimal open-loop control in problem (1.3). Then system (1.2) is stabilized at time τ by using the control $u^*(\tau) = u^0(\tau\,|\,x^*(\tau), \tilde{w}_\tau(\cdot))$. By construction, $u^*(\tau)$ is a functional of $u^*_\tau(\cdot), x^*_\tau(\cdot)$: $u^*(\tau) = u^0(t\,|\,x^*(\tau), \tilde{w}_\tau(\cdot))$, defined for all pairs (of generalized states) $\{u^*_\tau(\cdot), x^*_\tau(\cdot)\}$ for which problem (1.3) has a solution.

In this method of constructing stabilizing controls $u^*(\tau)$ we have not described the rules by which one proceeds from the available information $\{u^*_\tau(\cdot), x^*_\tau(\cdot)\}$ to the test perturbation $\tilde{w}_\tau(\cdot)$. These rules may be arbitrary, but the result of stabilization will be better, the more accurately the test signal $\tilde{w}_\tau(\cdot)$ can approximate the future perturbation $w^*(t), t \in T_\tau$. Clearly, this is possible only for fairly regular perturbations. This indicates the effective limits of the method. One way of using generalized states for optimal control problems was described in [6].

A device which, in every specific process, is capable of computing such controls $u^*(\tau), \tau \geq 0$, in real time will be called an optimal stabilizer (based on generalized states).

The purpose of this paper is to describe an algorithm for the operation of an optimal stabilizer and to point out some auxiliary optimal control problems that may be used, along with (1.3), to stabilize dynamical systems under persistent perturbations.

2. OPTIMAL STABILIZER OPERATING ALGORITHM

At the starting time $\tau = 0$, utilizing prior information, some signal $\tilde{w}_0(t) \in W, t \in T_\theta = [0, \theta[$ is chosen, and an open-loop solution of problem (1.3) is constructed. The optimal open-loop control $u^0(t\,|\,x_0, \tilde{w}_0(\cdot))$, $t \in T_\theta$, of problem (1.3) has the form [7]

$$u^0(t|x_0, \tilde{w}_0(\cdot)) = \rho\,\mathrm{sign}\,\Delta^0_0(t), \quad t \in T_\theta \tag{2.1}$$

where $\Delta^0_0(t) = -y'F(\theta - t)b, t \in T_\theta$, is an optimal co-control, y is an optimal n-vector of potentials, $F(t)$, $t \geq 0$, is the fundamental solution matrix of the system $\dot{x} = Ax$ and $\rho = y'F(\theta)x_0$ is the optimal (minimal) intensity of the control.

It is obvious from (2.1) that the computation at time τ of the control value $u^*(\tau) = u^0(\tau\,|x^*(\tau), \tilde{w}_\tau(\cdot))$ requires information on the parameters

$$t_1(\tau), \ldots, t_{p(\tau)}(\tau); \quad y(\tau); \quad \rho(\tau) \tag{2.2}$$

where $t_1(\tau) < \ldots < t_{p(\tau)}(\tau)$ are the switching points of the optimal open-loop control

$$u^0(t|x^*(\tau),\ \tilde{w}_\tau(\cdot)), \quad t \in T_\tau, \ (\Delta^0_\tau(t_i(\tau)) = 0, \ i = \overline{1, p(\tau)}).$$

It follows from the optimal conditions that the parameters (2.2) satisfy the system of equations

$$\begin{aligned} &\sum_{i=0}^{p(\tau)} \int_{t_i(\tau)}^{t_{i+1}(\tau)} F(\tau+\theta-t)bdtk_i\rho(\tau) + F(\theta)x^*(\tau) + \int_\tau^{\tau+\theta} F(\tau+\theta-t)q\tilde{w}_\tau(t)dt = 0 \\ &-y'(\tau)F(\tau+\theta-t_i(\tau))b = 0, \quad i = \overline{1, p(\tau)} \\ &y'(\tau)F(\theta)x^*(\tau) = \rho(\tau) \end{aligned} \tag{2.3}$$

where

$$t_0(\tau) = \tau, \ t_{p(\tau)+1}(\tau) = \tau + \theta, \ k_i = \mathrm{sign}\,\Delta^0_\tau(t_i(\tau)+0), \ i = \overline{0, p(\tau)}.$$

We shall refer to the equations of system (2.3) as the governing equations of the optimal stabilizer, and to the set $S(\tau) = \{p(\tau); k_i, i = \overline{0, p(\tau)}\}$ as the structure of the governing equations at time τ. Following [8], we can prove that under fairly general conditions the Jacobian of system (2.3) is non-singular over ranges with constant structure. To compute the elements (2.2) over ranges with constant structure at time τ, we can use the quantities $t_1(\tau - h), \ldots, t_{p(\tau)}(\tau - h)$: $y(\tau - h)$; $\rho(\tau - h)$, where h is the operating cycle of the optimal stabilizer, as an initial approximation. The details of the real-time numerical solution of systems like (2.3) over ranges with constant structure, as well as rules for moving from range to range, are analogous to those presented in [8].

Having solved the governing equations, one can form the control $u^*(\tau) = k_0\rho(\tau)$ and apply it to the input of system (2.2).

To investigate the properties of a dynamical system stabilized in the above manner, let us calculate the derivative $d\rho/d\tau$ in the case when the perturbation $w^*(\tau)$, $t \geq 0$, applied to the system is identical with the test signal $\tilde{w}_\tau(t)$, $t \geq \tau$, $\tau \geq 0$, included in the stabilization algorithm.

By (3.3), $d\rho/d\tau = -e'_{n+p}G^{-1}\partial f/\partial\tau$, where

$$e'_{n+p} = (\underbrace{0,\ldots,0}_{n+p-1}, 1)$$

G is the Jacobian of system (3.3)

$$G = \left\| \begin{array}{c:c:c} 2F(\tau+\theta-t_i(\tau))bk_i\rho, & 0 & \sum\limits_{i=0}^{p(\tau)} \int\limits_{t_i(\tau)}^{t_{i+1}(\tau)} F(\tau+\theta-t)bdtk_i \\ i=\overline{1,p(\tau)} & & \\ \hdashline \operatorname{diag}(y'F(\tau+\theta-t_i(\tau))Ab, & -b'F'(\tau+\theta-t_i(\tau)), & 0 \\ i=\overline{1,p(\tau)} & i=\overline{1,p(\tau)} & \\ \hdashline 0 & x^*(\tau)F'(\theta) & -1 \end{array} \right\|$$

and $\partial f/\partial\tau$ is the partial derivative of the left-hand side of system (2.3) with respect to τ.

It can be shown that

$$e'_{n+p}G^{-1} = (-y', \underbrace{0,\ldots,0}_{p})$$

and, consequently, $d\rho/d\tau = y'bk_{p(\tau)}\rho + y'q\tilde{w}_\tau(\tau + \theta)$.

Thus, the intensity $\rho = \rho(x^*(\tau))$ of the stabilizing control $u^*(\tau)$, $\tau \geq 0$, is a decreasing function ($d\rho/d\tau < 0$) in the region

$$\{x \in R^n:\ \rho(x) > y'q\tilde{w}_\tau(\tau+\theta)/|\Delta^0_\tau(\tau+\theta)|\}$$

where the norm of the state $x^*(\tau)$, $\tau \geq 0$, will also decrease

$$\|x^*(\tau)\| \leq \left(\left\| F^{-1}(\theta) \sum_{i=0}^{p(\tau)} \int\limits_{t_i(\tau)}^{t_{i+1}(\tau)} F(\tau+\theta-t)bdt \cdot k_i \right\| + \right.$$
$$\left. + \left\| F^{-1}(\theta) \int\limits_{\tau}^{\tau+\theta} F(\tau+\theta-t)y'bk_{p(\tau)}/(y'q)dt \right\| \right) \rho(x^*(\tau))$$

Remark. It is obvious from the above arguments that the minimum intensity $\rho(x)$ of the stabilizing control may be treated as a Lyapunov function satisfying the conditions of the theorem of asymptotic stability when $w^*(t) \equiv 0, t \geq 0$ [5, 9].

3. STABILIZATION WITH COMPENSATION OF TEST PERTURBATIONS

Along with θ, we will introduce another parameter $\theta_1 > 0$ such that $\theta = k^*\theta_1$, $k^* \in N$.

At a given time τ, consider the minimum interval $[l_*\theta_1, l^*\theta_1]$ that covers the interval T_τ, i.e. $\tau \in [l_*\theta_1, (l_* + 1)\theta_1[$, where

$$l^* = \begin{cases} l_* + k^*, & \text{if } \tau = l_*\theta_1 \\ l_* + k^* + 1, & \text{if } \tau > l_*\theta_1 \end{cases}$$

Using the times $l\theta_1$, $l = l_*, \ldots, l^* - 1$, we construct solutions $u^0_l(t \mid \tilde{w}_*(\cdot))$, $t \in [l\theta_1, (l+1)\theta_1[$ of the following problems of compensating the test perturbations

$$\begin{gathered} \rho_l \to \min, \quad \dot{x} = Ax + bu_l + q\tilde{w}_*, \quad x(l\theta_1) = 0 \\ x((l+1)\theta_1) = 0, \quad |u_l(t)| \leq \rho_l \\ t \in [l\theta_1, (l+1)\theta_1[, \quad l = l_*, \ldots, l^* - 1 \end{gathered} \tag{3.1}$$

where

$$\tilde{w}_*(t) = \begin{cases} \tilde{w}_{l_*\theta_1}(t), & t \in [l_*\theta_1, \tau[\\ \tilde{w}_\tau(t), & t \geq \tau \end{cases}$$

As a result, at any time τ we obtain a compensating control for the interval $T\tau$

$$\bar{u}_\tau(t) = \begin{cases} u^0_{l_*}(t \mid \tilde{w}_*(\cdot)), & t \in [\tau, (l_* + 1)\theta_1[\\ u^0_{l_*+1}(t \mid \tilde{w}_\tau(\cdot)), & t \in [(l_* + 1)\theta_1, (l_* + 2)\theta_1[\\ \ldots \\ u^0_{l^*-1}(t \mid \tilde{w}_\tau(\cdot)), & t \in [(l^* - 1)\theta_1, \tau + \theta[\end{cases}$$

Knowing the values of the control $\bar{u}_\tau(\cdot) = (\bar{u}_\tau(t), t \in T_\tau)$, we will construct a realization of a positional solution $\tilde{u}(\tau) = u^0(\tau \mid x^*(\tau), \bar{u}_\tau(\cdot), \tilde{w}_\tau(\cdot))$ of the optimal damping problem

$$\begin{gathered} \rho \to \min, \quad \dot{x} = Ax + b\tilde{u} + b\bar{u} + q\tilde{w}_\tau, \quad x(\tau) = x^*(\tau) \\ x(\tau + \theta) = 0, \quad |\tilde{u}(t)| \leq \rho, \quad t \in [\tau, \tau + \theta[\end{gathered} \tag{3.2}$$

In the same regime, the signal $u^*(\tau) = \bar{u}_\tau(\tau) + \tilde{u}(\tau)$ is applied at time τ to the input of system (2.2).

Remark. The problem considered above, of stabilization through compensation of the test signal, is a simplified version of the compensation problem. The problem in its full complexity may be written in the form

$$q_0 \to \min, \quad \dot{x} = Ax + bu + qw, \quad x(\tau) = x^*(\tau), \quad |x_j(t)| \leq q_0, \quad j = \overline{1,n}, \quad |u(t)| \leq 1, \quad t \in T_\tau \tag{3.3}$$

In mathematical terms, problem (3.3) is much more complicated than problem (3.1), (3.2), since it involves phase constraints. Although the method used here may be generalized to such problems too, the profit gained thereby hardly justifies the effort. It was therefore decided to confine attention to problem (3.1), (3.2), which approximates the main idea of compensation but leads to simpler constructions. The full compensation problem will be solved below, in Section 4, for a special case in which it becomes trivial.

4. A SPECIAL CASE OF THE STABILIZATION PROBLEM

Suppose that system (1.1) to be stabilized is subject to a bounded scalar perturbations $w(t)$, $t \geq 0$, and that its real motion is described by the equation

$$\dot{x} = Ax + bu + bw(t), \quad x(0) = x_0 \tag{4.1}$$

This situation arises, for example, if the control system is described by an nth order differential equation

$$x^{(n)}+a_1x^{(n-1)}+\ldots+a_nx=u+w(t)$$

In that case the algorithm described in Section 3 may be modified as follows.

Suppose that at time τ we have a known test perturbation $\tilde{w}_\tau(t)$, $\tau\in T_\theta$. As in Section 3, the control $u^*(\tau)$ applied at the input of system (4.1) may be expressed as the sum of a compensating control and a damping control: $u^*(\tau)=\bar{u}_\tau(\tau)+\tilde{u}(\tau)$. At time τ we assume that $\bar{u}_\tau(\tau)=-\tilde{w}_\tau(\tau)$. The damping control $\tilde{u}(\tau)$, $\tau\geq 0$ is constructed as a solution of the problem

$$\rho\to\min,\quad \dot{x}=Ax+b\tilde{u},\quad x(\tau)=x^*(\tau),\ x(\tau+\theta)=0,\quad |\tilde{u}(t)|\leq\rho,\quad t\in T_\theta$$

on the assumption that the influence of the perturbation $\tilde{w}_\tau(t)$, $\tau\in T_\theta$, is fully compensated.

The algorithm described may be supplemented by the following procedure. To compensate the effect over the interval T of the perturbation $\tilde{w}_\tau(\cdot)$, $\tau\leq\theta$, we construct a solution of the problem

$$\rho\to\min,\quad \dot{x}=Ax+b\bar{u}+b\tilde{w}_\tau,\quad x(\tau)=x^0(\tau),\ x(\theta)=0,\quad |\bar{u}(t)|\leq\rho,\quad t\in[\tau,\theta[$$

where $x^0(\tau)$, $\tau\in T_\theta$, $(x^0\,(0)=x_0)$ is a model trajectory, defined by the control $\bar{u}(\tau)=\bar{u}(\tau\,|\,x^0(\tau),\tilde{w}_\tau(\cdot))$ and the perturbation $\tilde{w}_\tau(\tau)$.

At each time τ, we calculate the deviation of the real trajectory $x^*(\tau)$, $\tau\in T_\theta$, of system (4.1) from the model trajectory $x^0(\tau)$: $x^*(\tau)-x^0(\tau)$. From the time $\bar{\tau}<\theta$ at which the action of the excluded perturbations $w^*(\tau)-\tilde{w}_\tau$, $\tau<\bar{\tau}$, produces a deviation $\|x^*(\bar{\tau})-x^0(\bar{\tau})\|\geq\varepsilon$ we begin to solve the problem

$$\rho\to\min,\quad \dot{x}=Ax+b\tilde{u},\quad x(\tau)=x^*(\tau)-x^0(\tau),$$

$$x(\tau+\theta)=0,\quad |\tilde{u}(t)|\leq\rho,\quad t\in T_\tau,\quad \tau\in[\bar{\tau},\theta[$$

where ε is a parameter characterizing the closeness of the real and model trajectories. The control $u^*(\tau)=\bar{u}_\tau(\tau)+\tilde{u}(\tau)$ is applied to the input of system (4.1). At time $\tau=\theta$, we proceed to the algorithm described previously in this section.

5. EXAMPLES

To illustrate the above results, let us apply them to the problem of stabilizing a mathematical pendulum in its upper, unstable equilibrium position, by a torque applied to the axis of the suspension. The torque is produced by a slave mechanism which acts as an integrating element. The slave mechanism, in turn, is subject to a certain controlling force u [10].

The linearized equation of motion of a mathematical pendulum is

$$\dot{x}_1=x_2,\quad \dot{x}_2=x_1+x_3,\quad \dot{x}_3=u$$

where $x_1=\varphi$ is the angle by which the pendulum deviates from the vertical, $x_2=\dot{\varphi}$ is the angular velocity, and x_3 is the torque applied to the pendulum. As a perturbed initial state $x(0)$ we take the point $x_0=(1.4,-1.25,-0.2)$.

Let us assume initially that, under the action of a persistent perturbation $w^*(t)$, $t\geq 0$, the dynamical system satisfies the equation

$$\dot{x}_1=x_2,\quad \dot{x}_2=x_1+x_3+w^*(t),\quad \dot{x}_3=u,\quad x(0)=x_0 \tag{5.1}$$

We put

$$w^*(t)=a_0(t)+a_1(t)\sin t+w_1(t),\quad t\geq 0 \tag{5.2}$$

where $a_0(t)=0.5(1+0.1\sin 0.3t)$, $a_1(t)=1+0.2\sin 0.2t$, $w_1(t)=0.1+0.1\cos 5t$. The value of the parameter θ is set equal to three.

Stabilization of system (5.1) by using a classical feedback-controlled damper [5] yields the trajectory $x^*{}_1(\tau)$, $\tau\geq 0$, of system (5.2) represented in Fig. 1 (curve 1).

When stabilization was performed according to the scheme of Section 2, the function

$$w_\tau(t)=a_0(\tau)+a_1(\tau)\sin t,\quad t\geq\tau,\quad \tau\geq 0 \tag{5.3}$$

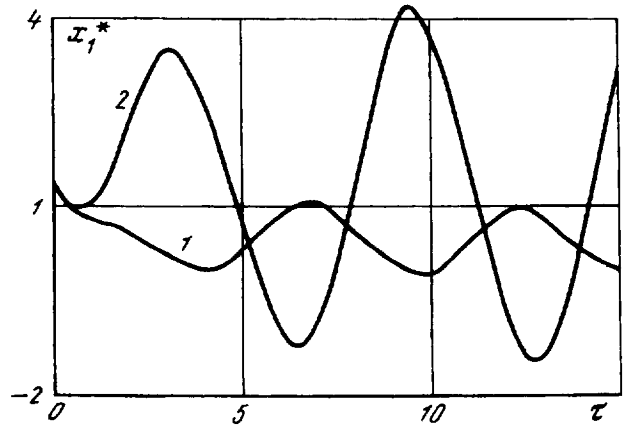

Fig. 1.

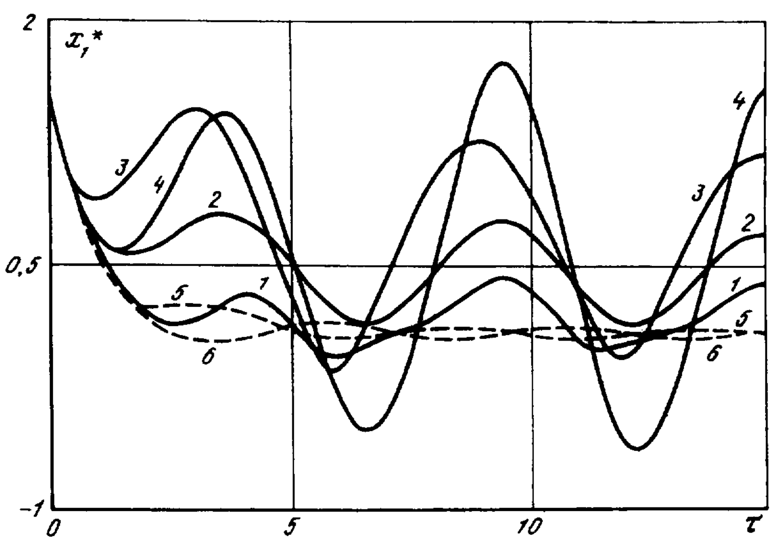

Fig. 2.

was used as a test perturbation. The resulting trajectory $(\tau), x_1^*(\tau)$, $\tau \geqslant 0$, is shown in Fig. 1 (curve 2).

Now let us consider the case in which the stabilized system, subject to persistent perturbations, is described by the equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + x_3, \quad \dot{x}_3 = u + w^*(t), \quad x(0) = x_0 \tag{5.4}$$

where the perturbations $w^*(t)$, $\tilde{w}_\tau(t)$, $t \geqslant \tau$, $\tau \geqslant 0$, are taken equal to the functions (5.2) and (5.3), respectively.

Curve 1 in Fig. 2 represents the trajectory $x_1^*(\tau)$, $\tau \geqslant 0$, of system (5.4) when stabilized according to the scheme of Section 3. Curve 2 is the result of stabilization by the Kalman–Letov method [10]. Curve 3 is the result of stabilization of system (5.4) according to the scheme of Section 3. Curve 4 was plotted using the damper of [5]. Curves 5 and 6 illustrate the stabilization of system (5.4) according to the scheme of Section 4. Curve 5 represents the trajectory $x_1^*(\tau)$, $\tau \geqslant 0$, obtained by using a single-phase stabilization procedure, and curve 6 represents the corresponding result with a two-phase procedure.

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