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## CONTROL THEORY

# Optimal Controller for a Time-Dependent System

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1. Let $T = [t_*, t^*]$, $0 \le t_* < t^* < +\infty$, $N$ be a positive integer, $h = \dfrac{t^* - t_*}{N}$, $A(t)$ and $b(t)$ $(t \in T)$ be a piecewise continuous $n \times n$ matrix function and a piecewise continuous $n$-vector function, and $T_u = \{t_*, t_* + h, \ldots, t^* - h\}$. A function $u(t)$ $(t \in T)$ is called a discrete control (with quantization step $h$) if $u(t) = u(t_* + kh)$, $t \in [t_* + kh, t_* + (k+1)h[$, $k = 0, 1, \ldots, N-1$.

In the class of discrete controls, we consider a family of linear problems depending on $\sup(\tau, z)$ [1]:

$$\begin{gathered} c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \\ x(\tau) = z, \quad Hx(t^*) = g, \\ |u(t)| \le 1, \quad t \in T(\tau) = [\tau, t^*], \quad x \in R^n, \\ u \in R, \quad g \in R^m, \quad \operatorname{rank} H = m < n. \end{gathered} \tag{1}$$

At $\tau = t_*$ and $z = x_0$, this family includes the original problem of optimizing a time-varying system.

Let $u^0(t|\tau, z)$, $t \in T(\tau)$, be the optimal open-loop control of problem (1) in the position $(\tau, z)$, and let $X(\tau)$ be the set of states $z$ at which problem (1) has a solution for a fixed $\tau$.

The function

$$u^0(\tau, z) = u^0(\tau|\tau, z), \quad z \in X(\tau), \quad \tau \in T_u \tag{2}$$

is called an optimal feedback control.

The control system of problem (1) supplemented with feedback (2) becomes a closed-loop system. Suppose that the behavior of the closed-loop system under persistent piecewise continuous disturbances $w(t)$ $(t \in T)$ is described by the equation

$$\dot{x} = A(t)x + b(t)u^0(t, x) + w(t), \quad x(t_*) = x_0. \tag{3}$$

The trajectory of system (3) is a solution to the equation

$$\begin{gathered} \dot{x} = A(t)x + b(t)u(t) + w(t), \\ x(t_*) = x_0, \quad u(t) = u^0(t_* + kh, x(t_* + kh)), \\ t \in [t_* + kh, t_* + (k+1)h[, \quad k = 0, 1, \ldots, N-1. \end{gathered}$$

Consider a particular control process with an initial state $x(t_*) = x_0^*$ and a disturbance $w^*(t)$, $t \in T$. They correspond to a trajectory $x^*(t)$ $(t \in T)$ of system (3) that satisfies the identity

$$\begin{gathered} \dot{x}^*(t) \equiv A(t)x^*(t) + b(t)u^0(t, x^*(t)) + w^*(t), \\ t \in T, \quad x(t_*) = x_0^*. \end{gathered}$$

We say that (i) $u^*(t) = u^0(t, x^*(t))$ $(t \in T_u)$ is a realization of the optimal feedback (2); (ii) $u^*(t)$ $(t \in T_u)$ is constructed in real time if the time required for calculating $u^*(\tau) = u^0(\tau, x^*(\tau))$ in every current position $(\tau, x^*(\tau))$ does not exceed $h$; (iii) this is done by a device called an optimal controller.

2. Assume that an operation algorithm for an optimal controller has been designed, and the optimal controller produced signals $u^*(t_*)$, $u^*(t_* + h)$, $\ldots$, $u^*(\tau)$ at the times $t_*$, $t_* + h$, $\ldots$, $\tau$, respectively. Let $x^*(\tau + h)$ be the system state at $\tau + h$ resulting from the action of these signals and the current disturbance $w^*(t)$, $t \in [t_*, \tau + h[$. According to (2), the open-loop solution $u^0(t|\tau + h, x^*(\tau + h))$, to problem (1) at $(\tau + h, x^*(\tau + h))$ is required for the controller to calculate $u^*(\tau + h)$. By assumption, the optimal controller has constructed $u^0(t|\tau, x^*(\tau))$ $[t \in T(\tau)]$ at the point $\tau$. A part of this solution $u^0(t|\tau, x^*(\tau))$, $t \in T(\tau + h)$, is the optimal open-loop control at $(\tau + h, x^0(\tau + h))$ resulting from the action of $u^0(\tau|\tau, x^*(\tau))$ at $(\tau, x^*(\tau))$ in the absence of any disturbance ($w(t) = 0$, $t \in [\tau, \tau + h[$). The actual state $x^*(\tau + h)$ is related to the ideal state $x^0(\tau + h)$ by

$$x^*(\tau + h) = x^0(\tau + h) + \int_\tau^{\tau + h} F(\tau + h)F^{-1}(s)w^*(s)ds.$$

For small $h > 0$ and bounded $w^*(t)$, $t \in [\tau, \tau + h[$, the vectors $x^*(\tau + h)$ and $x^0(\tau + h)$ are nearly identical; therefore, the controller can generate $u^0(t|\tau + h, x^*(\tau + h))$,

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$t \in T(\tau + h)$ by correcting the already known control $u^0(t|\tau, x^*(\tau))$, $t \in T(\tau + h)$. Below, we describe the basic steps of the correction procedure proposed.

In functional form, problem (1) for the position $(\tau, x^*(\tau))$ takes the form

$$\sum_{t \in T_u(\tau)} c_h(t)u(t) \to \max, \quad \sum_{t \in T_u(\tau)} d_h(t)u(t) = g(\tau),$$

$$|u(t)| \le 1, \quad t \in T_u(\tau);$$

$$c_h(t) = \int_t^{t+h} \psi'_c(\vartheta)b(\vartheta)d\vartheta, \tag{4}$$

$$d_h(t) = \int_t^{t+h} G(\vartheta)b(\vartheta)d\vartheta, \quad g(\tau) = g - G(\tau)x^*(\tau);$$

$T_u(\tau) = \{\tau, \tau + h, \ldots, t^* - h\}$, and $\psi_c(t)$ and $G(t)$ $[t \in T(\tau)]$ are the solutions to the equations

$$\dot{\psi} = -A'(t)\psi, \quad \psi(t^*) = c;$$
$$\dot{G} = -GA(t), \quad G(t^*) = H. \tag{5}$$

The set $T_{\sup}(\tau) = \{t_l, l = 1, 2, \ldots, m\} \subset T_u(\tau)$ is called a support if the matrix $D_{\sup}(\tau) = (d_h(t), t \in T_{\sup}(\tau))$ is nonsingular [2–4]. The pair $\{u_\tau(\cdot), T_{\sup}(\tau)\}$ consisting of the admissible control $u_\tau(\cdot) = (u(t), t \in T(\tau))$ and a support is called a support control.

The support $T_{\sup}(\tau)$ corresponds to the $m$-vector of potentials $\nu(\tau)$: $\nu'(\tau)D_{\sup}(\tau) = c'_{\sup}(\tau)$, $c_{\sup}(\tau) = (c_h(t), t \in T_{\sup}(\tau))$, the cocontrol

$$\Delta_h(t) = c_h(t) - \nu'(\tau)d_h(t)$$
$$= \int_t^{t+h} (\psi'_c(\vartheta) - \nu'(\tau)G(\vartheta))b(\vartheta)d\vartheta, \quad t \in T_u(\tau),$$

and the pseudocontrol $\omega(t)$, $t \in T_u(\tau)$:

$$\omega(t) = \begin{cases} -1 & \text{if } \Delta_h(t) < 0 \\ 1 & \text{if } \Delta_h(t) > 0; \end{cases}$$

$$\omega(t) \in [-1, 1] \text{ if } \Delta_h(t) = 0,$$
$$t \in T_n(\tau) = T_u(\tau) \backslash T_{\sup}(\tau),$$

The support values of $\omega(t)$ $[t \in T_{\sup}(\tau)]$ satisfy the equation

$$\sum_{t \in T_{\sup}(\tau)} d_h(t)\omega(t) + \sum_{t \in T_n(\tau)} d_h(t)\omega(t) = g(\tau).$$

The pseudocontrol $\omega^0(t)$ $[t \in T_u(\tau)]$ corresponding to the optimal support $T^0_{\sup}(\tau)$ is the optimal open-loop control $u^0(t|\tau, x^*(\tau))$, $t \in T(\tau)$, in problem (4) [3]. Thus, to generate the signal $u^*(\tau + h)$ at $\tau + h$, it is sufficient to know the optimal support $T^0_{\sup}(\tau + h)$. Let us describe the operations transforming the optimal support $T^0_{\sup}(\tau)$ into $T^0_{\sup}(\tau + h)$.

3. To simplify the presentation, we assume that the condition of dual nondegeneracy holds for any support $T_{\sup}(\tau)$, $\tau \in T_u$:

$$\Delta_h(t) \ne 0, \quad t \in T_n(\tau); \quad \Delta_h(t-h)\Delta_h(t+h) < 0$$
$$\text{if } \tau < t < t^* - h, \quad t \in T_{\sup}(\tau);$$

$$\Delta_h(\tau + h) \ne 0 \text{ if } \tau \in T_{\sup}(\tau);$$
$$\Delta_h(t^* - 2h) \ne 0 \text{ if } t^* - h \in T_{\sup}(\tau)$$

(the general case was analyzed in [3].) A point $t \in T_n(\tau)\backslash\tau$ is called a nonsupport zero if $\Delta_h(t-h)\Delta_h(t) < 0$. Let $T_{n0}(\tau)$ be the set of nonsupport zeros and $T_{sn}(\tau) = T_{\sup}(\tau) \cup T_{n0}(\tau) \cup \{\tau, t^*\} = \{t_k, k \in K(\tau) = k^*(\tau) + 1\}$, $K(\tau) = \{0, 1, \ldots, k^*(\tau)\}$. Denote by $T_k(\tau)$, $k \in K(\tau)$, the intervals where the cocontrol has a definite sign:

$$T_k(\tau) = \{t_{*k} = t_k, t_k + h, \ldots, t^*_k = t_{k+1} - h\}$$
$$\text{if } t_k \notin T_{\sup}(\tau);$$

$$T_k(\tau) = \{t_{*k} = t_k + h, t_k + 2h, \ldots, t^*_k = t_{k+1} - h\}$$
$$\text{if } t_k \in T_{\sup}(\tau).$$

If $t^* - h \in T_{\sup}(\tau)$, we set $T_{k^*}(\tau) = \varnothing$.

Assume that the following data have been obtained by solving problem (1) at position $(\tau, x^*(\tau))$ and have been stored:

(i) an optimal support $T^0_{\sup}(\tau)$;

(ii) the set $T_{n0}(\tau)$;

(iii) the support matrix $D_{\sup}(\tau)$ and the vector $d_h(\tau)$;

(iv) $G(t)$ and $\psi_c(t)$ at $t \in T_{on}(\tau)\backslash t^*$;

(v) $u^*(\tau)$, $\gamma(\tau) = u^*(\tau)$ if $\tau \notin T^0_{\sup}(\tau)$, and $\gamma(\tau) = \operatorname{sgn}\Delta^\tau_h(\tau + h)$ if $\tau \in T^0_{\sup}(\tau)$;

(vi) the vector

$$p(\tau) = \gamma(\tau)\sum_{k=0}^{k^*(\tau)} (-1)^k \sum_{t \in T_k(\tau)} d_h(t)$$
$$= \gamma(\tau)\sum_{k=0}^{k^*(\tau)} (-1)^k \int_{t_{*k}}^{t^*_k + h} G(\vartheta)b(\vartheta)d\vartheta;$$

(vii) the vector of potentials $\nu(\tau)$.

The problem to be solved by the controller at $\tau + h$ has the form

$$\sum_{t \in T_u(\tau+h)} c_h(t)u(t) \to \max,$$

$$\sum_{t \in T_u(\tau+h)} d_h(t)u(t) = g(\tau+h), \tag{6}$$

$$|u(t)| \le 1, \quad t \in T_u(\tau+h).$$

We calculate $u^*(\tau + h) = u^0(\tau + h|\tau + h, x^*(\tau + h))$ by using the data obtained at $\tau$. We distinguish the following two cases: (i) $\tau \notin T^0_{sup}(\tau)$, and (ii) $\tau \in T^0_{sup}(\tau)$.

In case (i), the initial values at $\tau + h$ are set to be $T_{sup}(\tau + h) := T^0_{sup}(\tau)$, $T_{n0}(\tau + h) := T_{n0}(\tau)$, $D_{sup}(\tau + h) := D_{sup}(\tau)$, and $\nu(\tau + h) := \nu(\tau)$. Integrating Eqs. (5) over $[\tau, \tau + h]$ with the initial conditions $\psi_c(\tau)$ and $G(\tau)$, we obtain and store $\psi_c(\tau + h)$ and $G(\tau + h)$; set $p(\tau + h) = p(\tau) - \gamma d_h(\tau)$; compute and store the vector $d_h(\tau + h)$; and set $\gamma(\tau + h) := \gamma(\tau)$ if $\tau + h \notin T_{sup}(\tau + h)$; $\gamma(\tau + h) := -\gamma(\tau)$ if $\tau + h \in T_{sup}(\tau + h)$; $T_{n0}(\tau + h) = T_{n0}(\tau)$ if $\tau + h \notin T_{n0}(\tau)$; $T_{n0}(\tau + h) = T_{n0}(\tau)\backslash\{\tau + h\}$ if $\tau + h \in T_{n0}(\tau)$; $D_{sup}(\tau + h) = D_{sup}(\tau)$, and $\nu(\tau + h) = \nu(\tau)$. If $\tau + h \notin T_{sup}(\tau)$, then $T_{sup}(\tau + h) = T_{sup}(\tau)\backslash\tau \cup \tau + h$ and $k^*(\tau + h) = k^*(\tau)$. If $\tau + h \in T_{sup}(\tau)$, then $T_{sup}(\tau + h) = T_{sup}(\tau)\backslash\tau$ and $k^*(\tau + h) = k^*(\tau) - 1$.

The iteration process transforming the current support $T_{sup}(\tau + h)$ is performed by steps. For simplicity, we set $T_{sup} = T_{sup}(\tau + h)$, $T_n = T_n(\tau + h)$, $T_{sup} = T_{sup}(\tau + h)$, $T_{n0} = T_{n0}(\tau + h)$, $D_{sup} = D_{sup}(\tau + h)$, $\nu = \nu(\tau + h)$, $p = p(\tau + h)$, $T_k = T_k(\tau + h)$, $k^* = k^*(\tau + h)$, and $\gamma = \gamma(\tau + h)$.

**Preliminary step.** From the equation $D_{sup}\omega_{sup} = g(\tau + h) - p$, we find $\omega(t)$, $t \in T_{sup}$. If

$$|\omega(t)| \le 1, \quad t \in T_{sup}, \tag{7}$$

then we set $T^0_{sup}(\tau + h) = T_{sup}$, feed the signal $u^*(\tau + h) = \gamma(\tau + h)$ if $\tau + h \notin T^0_{sup}$; or $u^*(\tau + h) = \omega(\tau + h)$ if $\tau + h \in T^0_{sup}$ as input to system (3), and go to the position $(\tau + 2h, x^*(\tau + 2h))$.

If (7) does not hold, then we choose $t^0$ from $T_{sup}$ such that $|\omega(t^0)| = \max|\omega(t)|$ at $t \in T_{sup}$ and find the vector $\Delta\nu$: $-D'_{sup}\Delta\nu = (\Delta\delta_h(t),\ t \in T_{sup})$, where $\Delta\delta_h(t^0) = \operatorname{sgn}\omega(t^0)$ and $\Delta\delta_h(t) = 0$, $t \in T_{sup}\backslash t^0$.

Define the function

$$\delta_h(t, \sigma) = \Delta_h(t) + \sigma\Delta\delta_h(t), \quad t \in T_u(\tau + h),$$
$$\sigma \ge 0 \quad (\text{varied cocontrol}), \tag{8}$$

where

$$\Delta\delta_h(t) = -\Delta\nu' d_h(t) = -\int_t^{t+h} \Delta\nu' G(\vartheta)b(\vartheta)d\vartheta,$$
$$t \in T_n$$

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