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SYNTHESIS OF OPTIMAL CLOSED SYSTEMS[1]

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Synthesis of optimal feedback, feedforward, and direct and combined coupling for nonlinear dynamic systems is considered in the paper. The method proposed is based on two procedures: a piecewise-linear approximation of the initial nonlinear problem and an asymptotic correction of the solution of a piecewise-linear problem. In its turn, the piecewise linear problem solution method is based on solution of a linear problem of optimal control and optimization for a finite parameter number. The results are illustrated by examples.

Keywords: *synthesis of optimal control systems, piecewise linear approximation, asymptotic correction of a solution.*

INTRODUCTION

The paper presents a new approach to the problem of synthesis of optimal control systems. As is generally known, in the optimal control (OC) theory, by the synthesis of optimal systems is meant construction of optimal feedback, feedforward, and combined loops. Among them, feedback is the most universal and widely used in practice. The problem of synthesis of optimal systems is central in the control theory and remains unsolved till now even for linear systems with geometrical constraints on control.

The latest results obtained in Minsk on synthesis of a state feedback-type are connected with the following problem of OC of the nonlinear system

$$c' x(t^*) \to \max, \quad \dot{x} = f(x) + bu, \quad x(0) = x_0, \tag{1}$$

$$Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T = [0, t^*].$$

Here $x = x(t)$ is an n-vector of a system state at the instant t, $u = u(t)$ is the value of scalar control, $f(x)$, $x \in X \subset R^n$, is a function differentiable a sufficient number of times, $g \in R^m$, and rank $H = m < n$.

In the OC theory, the function

$$u = u^0(t, x), \ |u^0(t, x)| \le 1, \ x \in X_t, \ t \in T,$$

with the sets of definition X_t, $t \in T$, is called (classical) optimal feedback state if the control system $\dot{x} = f(x) + bu^0(t, x)$, $x(\tau) = z$, closed by the feedback moves along an optimal path $x^0(t \mid \tau, z)$, $t \in [\tau, t^*]$, from any position (τ, z).

The problem of synthesis of optimal systems was a trigger in appearance of the OC theory. It was formulated by experts in automatic control in the late 40s of the 20th century in connection with the necessity of creation at that time of

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control systems with high-quality transients. Though there were no analogs of such problem in mathematics of those years, engineers have solved it for simple systems. To solve it, they, following traditions of the theory of automatic control, first opened the control system and obtained for the open system an OC of the programmed type $u = u^0(t), t \in T$, which played an auxiliary role and was not used in actual control. At the second stage, the programmed solution was closed, i.e., an optimal feedback was constructed, which could be used for closure of a real control system. A group of mathematicians guided by L. S. Pontryagin has selected from this scheme the problem of constructing optimal programmed control as the basic problem of OC theory. Such problem is much easier than the problem of synthesis of optimal systems and is close in its form to problems of classical calculus of variations. This can explain the fact that the main results achieved up to now in OC theory involve only programmed solutions.

Optimal systems can be synthesized using Pontryagin's maximum principle (a fundamental result of the mathematical theory of optimal processes) [1] only for two-dimensional systems (except for some individual exceptions of higher order). Hopes for Bellman's dynamic programming (DP) (the second fundamental result of the OC theory) [2] were not justified. It is well known that, basically, it is aimed at constructing a feedback-type OC. However, the "damnation of dimension" does not allow realizing efficiently the DP scheme for systems of the order higher than the second one, despite its successful mathematical substantiation in the last few years. Failures of DP in synthesis of optimal systems are caused by its extreme generality. The DP method allows one to study a very wide class of processes. The processes may make big jumps at any moment. There is no time to make any calculations for them in the course of control. Therefore, solution of the problem should be prepared in advance prior to the beginning of the process of real control, which requires both big time expenditures and big size of RAM. From the modern point of view, the shortage of the DP approach consists also in the fact that the whole work (as well as in the case of optimal programmed control) is to be carried out prior to the beginning of the control process, and there is no need to perform any operation, i.e., real control can be implemented not involving extremely powerful modern computers. As a result, such a "lazy" control turns out to be unrealizable. Meanwhile, the majority of real processes is continuous and has finite rates of development that allows additional calculations to be carried out during control. Unfortunately, this circumstance cannot be taken into account within the framework of DP.

The main idea of our approach to the problem of optimal system synthesis consists in additional calculations during control of processes having a limited rate of development. It turned out later that the approach described below could be generalized to the case of discontinuous transients generated by impulse disturbances.

The latest works of the authors of [3–10] devoted to the new approach to optimal systems synthesis are presented in this paper.

1. THE SCHEME OF THE PROPOSED APPROACH

Let us return to problem (1). We will first present the scheme of its solution we propose. It contains two procedures: 1) synthesis of OC for piecewise linear approximation of problem (1) and 2) asymptotic correction of a positional solution of the piecewise linear problem.

To construct a piecewise linear approximation, we will divide the domain $X \subset R^n$, in which processes of problem (1) take place, into the convex polyhedral sets X_i, $i = \overline{1, p}$:

$$X = \bigcup_{i=1}^{P} X_i, \; X_i \cap X_j = \varnothing, \; i \neq j.$$

As a piecewise linear approximation of the function $f(x)$, $x \in X$, we will introduce the function $\hat{f}(x)$, $x \in X$, continuous on X and linear on each set X_i, $i = \overline{1, p}$:

$$\hat{f}(x) = \hat{f}_i(x), \quad x \in X_i, \quad i = \overline{1, p},$$

$$\hat{f}_i(x) = A_i x + a_i, \quad x \in X_i, \quad i = \overline{1, p}.$$

By approximation accuracy we will mean the number

$$\delta = \max_{x \in X} \; || f(x) - \hat{f}(x) || \, / \, || f(x) ||. \tag{2}$$

The approximation accuracy can be made arbitrarily high if we select a sufficiently large p.

The problem

$$c' x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + bu, \quad x(0) = x_0, \tag{3}$$

$$H x(t^*) = g, \quad |u(t)| \le 1, \quad t \in T,$$

is called the piecewise linear approximation of problem (1).

Optimal control in problem (3) is constructed using solution of a linear problem with intermediate phase constraints and optimization at the moments of transition between domains of linearity of the function $\hat{f}(x)$, $x \in X$.

The algorithm of OC synthesis in a linear problem with intermediate phase constraints is a generalization of the algorithm of OC synthesis in the linear problem with terminal constraints

$$c' x(t^*) \to \max, \quad \dot{x} = Ax + bu, \quad x(0) = x_0, \tag{4}$$

$$Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T.$$

The second procedure (asymptotic correction of a solution of piecewise linear problem (3)) in solving nonlinear problem (1) is as follows. Problem (1) is written in the equivalent form

$$c' x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + \delta g(x) + bu, \quad x(0) = x_0, \tag{5}$$

$$Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T,$$

where $g(x) = (f(x) - \hat{f}(x)) / \delta$.

Then problem (5) is immersed in the set of problems

$$c' x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + \mu g(x) + bu, \quad x(0) = x_0 \tag{6}$$

$$Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T,$$

depending on a small parameter μ.

To construct in problem (6) the asymptotic solution $u^s(t, x, \mu)$, $x \in X_t$, $t \in T$, $\mu \ge 0$, of an arbitrary degree of accuracy s, fast algorithms are developed based on solution of linear and piecewise linear problems (3) and (4).

As a solution $u^0(t, x)$, $x \in X_t$, $t \in T$, of problem (1), the function $u^s(t, x, \delta)$, $x \in X_t$, $t \in T$, is taken.

It follows from the proposed scheme of synthesis of optimal system (1) that the scheme is based on OC synthesis in linear problem (4). We will dwell on the latter problem in more detail.

2. SYNTHESIS OF OPTIMAL FEEDBACK FOR LINEAR SYSTEMS

First, let us specify the class of admissible controls. Since the above-mentioned problems cannot be solved without a discrete-operation computer, we will pass from the piecewise continuous controls to discrete ones,

$$u(t) = u(kh), \quad t \in [kh, (k+1)h[, \quad k = \overline{0, N-1},$$

where $h = t^* / N$ is a time sampling (discretization) period. From a constructive point of view, such a replacement is quite natural and piecewise continuous and measurable controls are natural in the qualitative OC theory.

To introduce the concept of optimal feedback, we will immerse problem (4) into the set of problems

$$c' x(t^*) \to \max, \quad \dot{x} = Ax + bu, \quad x(\tau) = z, \tag{7}$$

$$Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T^\tau = [\tau, t^*],$$

depending on a scalar $\tau \in T_h = \{0, h, \dots, (N-1)h\}$ and an n-vector z.

Let $u^0(t|\tau, z), t \in T^\tau$, be an optimal programmed control of problem (7) for a position (τ, z); X_τ be a set of all initial states $z \in R^n$ for which solution of problem (7) at the instant τ exists.

The function

$$u^0(\tau, x) = u^0(\tau|\tau, x), \quad x \in X_\tau, \quad \tau \in T_h, \tag{8}$$

is called OC of the type of a classical (discrete) feedback of problem (4).

Classical formulation of the problem of optimal synthesis is related to determinate systems and consists in constructing an explicit formula for (8), whereas DP requires its tabulation. Such requirements cannot be satisfied even for linear problem (4). To overcome the deadlock, we will analyze what we actually need for the real control [11].

Let function (8) be constructed. We will use it to close a real system for which system (4) is a mathematical model. There is no point in closing mathematical model since it will suffice for it to use optimal programmed control. Feedbacks (8) are necessary to close a real system because behavior of a real system differs from behavior of a mathematical model due to errors in simulation and effect of unknown external disturbances. We will describe behavior of a real closed system as

$$\dot{x} = Ax + bu^0(t, x) + w, \quad x(0) = x_0, \tag{9}$$

where w is a population of terms reflecting simulation inaccuracy and external disturbances. For brevity sake let us call w disturbance.

By a path of system (9), we will mean solution of the equation

$$\dot{x} = Ax + bu(t) + w, \quad x(0) = x_0,$$

with the control

$$u(t) = u^0(kh, x(kh)), \ t \in [kh, (k+1)h[, \ k = \overline{0, N-1}.$$

Clearly, such a solution always exists. If we do not pass to discrete controls, then serious mathematical difficulties arise in solving closed system (9) since the right-hand side of differential equation (9) is discontinuous in x.

Let us consider a particular control process, in which an initial state x_0^* and disturbance $w^*(t), t \in T$, are realized. The latter will generate in (9) a transient $x^*(t)$, $t \in T$, along which the identity

$$\dot{x}^*(t) \equiv Ax^*(t) + bu^0(t, x^*(t)) + w^*(t), \quad t \in T, \quad x(0) = x_0^* \tag{10}$$

is fulfilled. It can be seen from (10) that feedback (8) as a whole is not used in a concrete control process, only its values

$$u^*(t) = u^0(t, x^*(t)), \quad t \in T_h, \tag{11}$$

along an isolated continuous curve $x^*(t)$, $t \in T$ are necessary. Moreover, it will suffice to know at each current instant of time $\tau \in T_h$ how to calculate the current value $u^*(\tau) = u^0(\tau, x^*(\tau))$ in a time not exceeding h (i.e., in real time) knowing the current state $x^*(\tau)$. Note that it is more difficult to define the concept of real time conditions in the case of piecewise continuous controls.

We will call function (11) the realization of an optimal feedback in a particular control process, and a device capable to calculate its values in real time in each specific control process (10) by the optimal controller.

Thus, the problem of synthesis of an optimal control system was reduced to construction of an algorithm of optimal controller operation.

Algorithm of operation of an optimal controller is based on a special dual method, which very quickly calculates current values of optimal feedback. Details of the algorithm are published in [4]. We will present only its basic concepts.

The above-mentioned dual algorithm deals with supports of problem (4), which are defined as follows.

Let us single out from T_h a set of m elements $T_{on} = \{t_l, \ l = \overline{1, m}\}$. Let $\chi_l(s)$, $s \in T$, $l = \overline{1, m}$, be solutions of the equation

$$\dot{x} = Ax + bu$$

with the following initial condition and controls:

$$x(0)=0,\ u(t)=\begin{cases}1, & t\in[t_l, t_l+h[;\\ 0, & t\in T\setminus[t_l, t_l+h[,\quad t_l\in T_{on},\quad l=\overline{1,m}.\end{cases}$$

Let us form a matrix

$$D_{on}=\left(H_{\chi l}(t^*),\quad l=\overline{1,m}\quad\right). \tag{12}$$

We will call the set T_{on} a support of problem (4) if $\det D_{on}\neq 0$. A support characterizes controllability of the output signal $z=Hx(t^*)$ with the help of impulses applied at basic moments $t\in T_{on}$.

We will associate a support T_{on} with the following (accompanying) elements:

1) a vector of Lagrangian multipliers $\nu=\nu(I)$, $I=\{1,2,\dots,m\}$, being a solution of the equation

$$\nu' D_{on}=c'_{on},\quad c_{on}=(c(t),\ t\in T_{on}), \tag{13}$$

2) a co-path $\psi(t)$, $t\in T$, which is a solution of the conjugate equation

$$\dot\psi=-A'\psi,\quad \psi(t^*)=c-H'\nu, \tag{14}$$

3) a co-control

$$\Delta(t)=\int_t^{t+h}\psi'(\theta)\,b\,d\theta,\quad t\in T_h, \tag{15}$$

4) a pseudo-control $\omega(t), t\in T$. Non-support values $\omega(t), t\in T_n = T_h\setminus T_{on}$, of a pseudo-control are defined as

$$\omega(t)=-1\ \text{if}\ \Delta(t)<0,\quad \omega(t)=1\ \text{if}\ \Delta(t)>0,$$

$$\omega(t)\in[-1,1]\ \text{if}\ \Delta(t)=0,\ t\in T_n.$$

We will find support values $\omega(t)$, $t\in T_{on}$, from the equation

$$\sum_{t\in T_{on}} d(t)\,\omega(t)+\sum_{t\in T_n} d(t)\,\omega(t)=g_0.$$

THEOREM. If $|\omega(t)|\leq 1, t\in T_{on}$, then $\omega(t), t\in T$, is an OC of problem (4). The appropriate support T_{on} is called the optimal support of problem (4). Problem (4) has an optimal support if this problem has admissible controls.

Therefore, to construct optimal programmed control, it will suffice to know Lagrange vector and zeroes of the co-control accompanying the optimal support.

The dual method allows one to construct the optimal support T^0_{on} from any support T_{on} in a finite number of iterations. Each iteration of the method is a transition of zeroes of a co-control in definite directions up to a complete relaxation of the dual objective function of problem (4). Dynamic implementation of the dual method is described in [4]. It is based on the following:

1) an analysis of behavior of zeroes of co-control, allowing one to select from T_h a set of potentially possible new zeroes and to avoid computing expenditures for handling certainly unpromising points,

2) storage and conversion during iterations of a small volume of auxiliary information for fast calculation of necessary values accompanying a current support of elements.

Algorithm of operation of an optimal controller. Prior to the beginning of control process, we construct an optimal support $T^0_{on}(0)$ for the position $(0, x^*_0)$. To do this, we solve problem (4). Since this operation can be done before the beginning of the control process, time expenditures at this stage do not play any role.

Following definition (11), we put at the initial moment $\tau=0$ that $u^*(0)=u^0(0,x^*_0)$.

Let an optimal controller have operated on the interval $[0,\tau[$, $\tau\in T_h$, and the control process be in the position $(\tau, x^*(\tau))$, $x^*(\tau)\in X_\tau$, $\tau\in T_h$. To calculate the signal $u^*(\tau)$, the controller should solve problem (7) for the state $z=x^*(\tau)$.

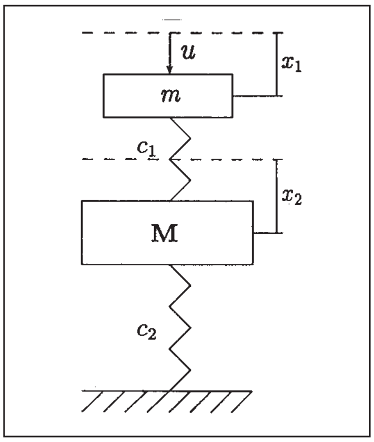

Fig. 1

It is generally known that main time expenditures in solving OC problems are associated with integration of the direct and conjugate systems. As can be seen from the aforesaid, speed of an algorithm plays the main role in implementation of an optimal feedback. In this connection, it is advisable to estimate efficiency of the method not by the amount of iterations (whose number differs in different methods) but by a summarized length of intervals on which the direct or conjugate systems are integrated for construction of OC. Numerical experiments may give a general idea of efficiency of the method.

The system reached the state $x^*(\tau)$ from the state $x^*(\tau-h)$ under the control $u^*(t)=u^*(\tau-h)$, $t\in[\tau-h,\tau[$, and disturbance $w^*(t)$, $t\in[\tau-h,\tau[$. By the assumption, the optimal controller has solved problem (7) at the previous moment $\tau-h$ for the position $(\tau-h, x^*(\tau-h))$. We will designate by $x^0(t)=x^0(t\,|\,\tau-h, x^*(\tau-h))$, $t\in T^{\tau-h}$, an optimal path of problem (7) for a position $(\tau-h, x^*(\tau-h))$.

It is easy to show that

$$x^*(\tau)=x^0(\tau)+\int_{\tau-h}^{\tau} F(\tau)F^{-1}(s)\,w^*(s)\,ds \ \ (\dot F=AF, \quad F(0)=E),$$

i.e., the states $x^*(\tau)$, $x^0(\tau)$ under a limited $w^*(s)$, $s\in[\tau-h,\tau[$ are the closer to each other the less h. Boundedness of $w^*(t)$, $t\in T$, together with boundedness of $u^*(t)$, $t\in T$, ensure a limited rate of development of the process $x^*(t)$, $t\in T$, that is immediately used in the stated approach. At small h, appropriate zeroes of the co-controls $\Delta(t\,|\,\tau-h, x^*(\tau-h))$, $t\in T^{\tau-h}$, $\Delta(t\,|\,\tau, x^*(\tau))$, $t\in T^{\tau}$, constructed using the supports $T^0_{on}(\tau-h)$ and $T^0_{on}(\tau)$. Therefore, distances for which zeroes of the first co-control should be shifted up to coincidence with zeroes of the second co-control are very small. It is they that define time intervals on which the conjugate system should be integrated in the passage from $x^*(\tau-h)$ to $x^*(\tau)$. And as is noted above, the rate at which the dual method creates an optimal support $T^0_{on}(\tau)$ by adjusting the support $T^0_{on}(\tau-h)$ depends on the interval of integration.

Example 1. Let us illustrate the efficiency of the proposed approach by the example of damping, in a fixed time and with a minimum fuel consumption, oscillations in a two-mass system such as a quarter model of an automobile (Fig. 1).

Mathematical model of the problem is as follows:

$$\int_0^{25} u(t)\,dt \to \min, \quad \dot x_1 = x_3, \quad \dot x_2 = x_4,$$

$$\dot x_3 = -x_1 + x_2 + u, \quad \dot x_4 = 0{,}1\,x_1 + 1{,}02\,x_2, \tag{16}$$

$$x_1(0)=x_2(0)=0, \quad x_3(0)=2, \quad x_4(0)=1,$$

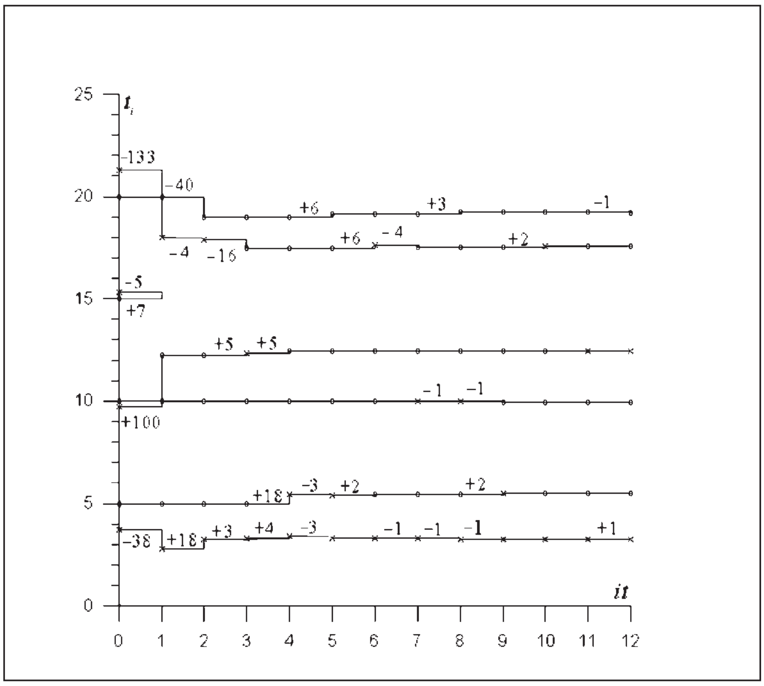

Fig. 2

$$x_1(25) = x_2(25) = x_3(25) = x_4(25) = 0, \quad 0 \le u(t) \le 1, \quad t \in [0, 25[,$$

where $x_1 = x_1(t)$ is deviation of the upper mass from equilibrium position, $x_2 = x_2(t)$ is deviation of the lower mass from equilibrium position, $x_3 = dx_1 / dt$, $x_4 = dx_2 / dt$, $u = u(t)$ is fuel consumption per second at the instant of time t.

Let the considered oscillatory system undergo a disturbance during control and, as a result, behavior of the system can be described by the equations

$$\dot{x}_1 = x_3, \; \dot{x}_2 = x_4, \; \dot{x}_3 = -x_1 + x_2 + u, \; \dot{x}_4 = 0,1x_1 + 1,02x_2 + w,$$

and the realized disturbance has the form

$$w^*(t) = 0.3 \sin 4t, \quad t \in [0, 9.75[; \quad w^*(t) \equiv 0, \quad t \ge 9.75.$$

This disturbance of optimal controller is not known, but a current state $x^*(\tau)$ is accessible for it at each moment $\tau \in T_h$.

Following the presented approach, a preliminary work has been done (before the beginning of the control process) in constructing an optimal support $T_{on}^0(0)$ in problem (16) in order to calculate the initial value $u^*(0)$ of the realization of an optimal feedback. Though time spent for construction of the initial optimal support $T_{on}^0(0)$ does not play a basic role in solving the problem of synthesis of an optimal system, we will present some numerical results illustrating efficiency of the dual method, which will be used hereafter for operation of an optimal controller.

Let us first describe solution of the problem for the sampling period $h = 0.025$. As an initial support, the population of the moments $T_{on} = \{5, 10, 15, 20\}$ uniformly distributed over the control interval $T = [0, 25]$ is selected. It is associated with the set of non-support zeroes [4] of a co-control consisting of four units $\{3.725;\; 9.725;\; 15.3;\; 21.3\}$.

Figure 2 presents transitions of zeroes of co-control during iterations. The numbers specify transition of each zero during one iteration in units of h. The best value of performance criterion is 6.331252.

One complete integration of a direct and a conjugate system on the interval T is taken as a unit of complexity of the method. It is assumed here that enough microprocessors are available for parallel calculations. One integration is necessary to construct the basic matrix D_{on} and one to reveal non-basic zeroes of co-control. Complexity of iterations of the dual method is 0.433.

TABLE 1

h	**Performance criterion**	**Complexity**
0.25	6.353339	0.64
0.0025	6.330941	0.4149
0.001	6.330938	0.39

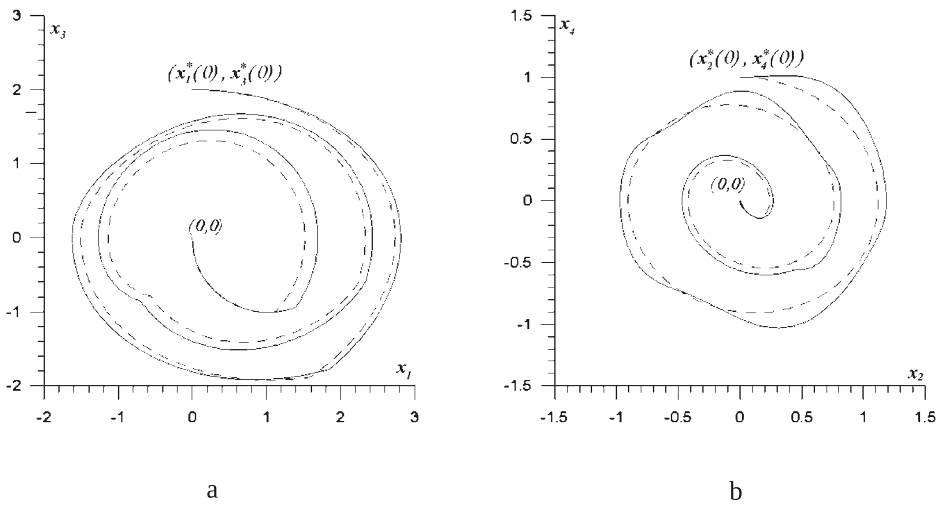

Fig. 3

Table 1 presents some information on solution of problem (16) for other sampling periods. It can be seen from the Table that it will suffice for construction of optimal programmed control to integrate the direct and conjugate systems on intervals whose total length does not exceed 65% of the control interval. (This number can be reduced to 40 if small h are used in the first two cases for trial steps.) Note that several tens of complete integrations of a system are necessary for similar problems in the existing methods of constructing OC [12].

Figure 3 shows projections on the planes x_1x_3 and x_2x_4 of transients in system (16) closed by an optimal feedback. Dashed lines correspond to optimal program paths and continuous lines to the paths obtained under control $u^*(\tau)$, $\tau \in T$, and disturbance $w^*(\tau)$, $\tau \in T$.

Dynamics of zeroes of co-control it presented in Fig. 4a, which can be used to determine optimal programmed control $u^0(t \mid x^*(\tau))$, $t \in T^\tau$, for any position $(\tau, x^*(\tau))$, $\tau \in T_h$. Realization of $u^*(t)$, $t \in T$, is given in Fig. 4b. The realized support values of control are marked by points. Figure 4c shows complexity $A(\tau)$ of correction of current supports by the dual method at each moment $\tau \in T_h : A(\tau) \le 0.02$.

Let α be the time required by a microprocessor to integrate the system over the interval T. When such microprocessors are used, feedback signal should be worked out in time $0.02\alpha \le h$. Based on this inequality, the sampling period h can be selected for available microprocessors and, on the other hand, appropriate microprocessors can be found for the given h. Many modern microprocessors can perform this operation in time $h = 0.025$ if real time in problem (16) is measured, for example, in seconds.

Remark. It was assumed till now that the initial state x_0^* of the control system is known beforehand, prior to the beginning of the control process. From the practical point of view, the case is interesting when the initial state is not known beforehand but belongs to a limited set. This state is known at the moment $\tau = 0$, when the process of real control starts. The presented constructions can also be generalized to this case.

3. OPTIMAL CONTROL OF A LINEAR SYSTEM WITH INTERMEDIATE PHASE CONSTRAINTS

Problems of OC with intermediate phase constraints are frequently used [12] in solving problems in which phase constraints are imposed on the whole control interval. On the other hand, they arise also as auxiliary ones in OC problems for nonlinear systems without phase constraints. Therefore, efficient algorithms of their programmed and positional solution are of a big applied value. The approach described in Section 2 to solution of linear OC problems with terminal constraints is generalized in the present section to a new more complicated class of OC problems.

Idea of the Algorithm. Let us consider in a class of discrete controls an OC problem with intermediate phase constraints

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \tag{17}$$

$$g_*(s) \le H(s)x(s) \le g^*(s), \quad s \in S; \quad |u(t)| \le 1, \quad t \in T.$$

Here $A(t)$, $b(t)$, and $t \in T$, are piecewise continuous $n \times n$ matrix and n vector functions, $g_*(s)$, $g^*(s) \in R^{m(s)}$, $H(s) \in R^{m(s) \times n}$; $s \in S$; $H_{(i)}(s)$ is the ith row of matrix $H(s)$, $i \in I(s) = \{1, \dots, m(s)\}$, $m = \sum_{s \in S} m(s)$, $S = \{s_1, \dots, s_{j^*}\} \subset T_h \cup t^*$ is a set of intermediate elements: $t_* + h < s_1 < \dots < s_{j^*} = t^*$ $T(s) = [t_*, s[$, $s \in S$; $J = \{1, \dots, j^*\}$.

Concepts of an admissible $u(\cdot) = (u(t), t \in T)$ and an optimal $u^0(\cdot)$ programmed controls are introduced as usual. Each admissible control $u(\cdot)$ is associated with a path $x(t)$, $t \in T$, and an output signal $z(s) = H(s)x(s)$, $s \in S$, of control system (17).

To introduce the concept of a support of problem (17), we will write the latter in the equivalent functional form

$$\sum_{t \in T_h} c(t)u(t) \to \max, \quad \tilde{g}_*(s) \le \sum_{t \in T_h} d(s,t)u(t) \le \tilde{g}^*(s), \quad s \in S, \ |u(t)| \le 1, \quad t \in T_h.$$

Here,

$$c(t) = \int_t^{t+h} \psi_c'(\vartheta) b(\vartheta)\, d\vartheta,$$

$$d(s,t) = \begin{pmatrix} d_i(s,t), \\ i \in I(s) \end{pmatrix} = \begin{cases} \int_t^{t+h} G(s,\vartheta) b(\vartheta)\, d\vartheta, & s > t, \\ 0, & s \le t, \end{cases}$$

$$\tilde{g}_*(s) = g_*(s) - G(s, t_*)x_0, \ \tilde{g}^*(s) = g^*(s) - G(s,t)x_0,$$

$\psi_c(t)$, $t \in T$, is the solution of the conjugate equation

$$\dot{\psi} = -A'\psi, \quad \psi(t^*) = c,$$

$G(s,t)$, $t \in T(s)$, is an $(m(s) \times n)$ matrix function being a solution of the equation

$$\dot{G} = -GA(t), \ G(s,s) = H(s).$$

Following [13], we will select for each $s \in S$ a subset $I_{on}(s)$ from $I(s)$. Designate $S_{on} = \{s \in S : I_{on}(s) \ne \varnothing\}$ and $I_{on} = \{I_{on}(s), \ s \in S_{on}\}$. From the set T_h, we will select a subset T_{on} so that $|T_{on}| = \sum_{s \in S_{on}} |I_{on}(s)|$. We will form a matrix

$$D_{on} = D(I_{on}, \ T_{on}) = \begin{pmatrix} d_i(s,t), & t \in T_{on}, \\ i \in I_{on}(s), & s \in S_{on} \end{pmatrix}.$$

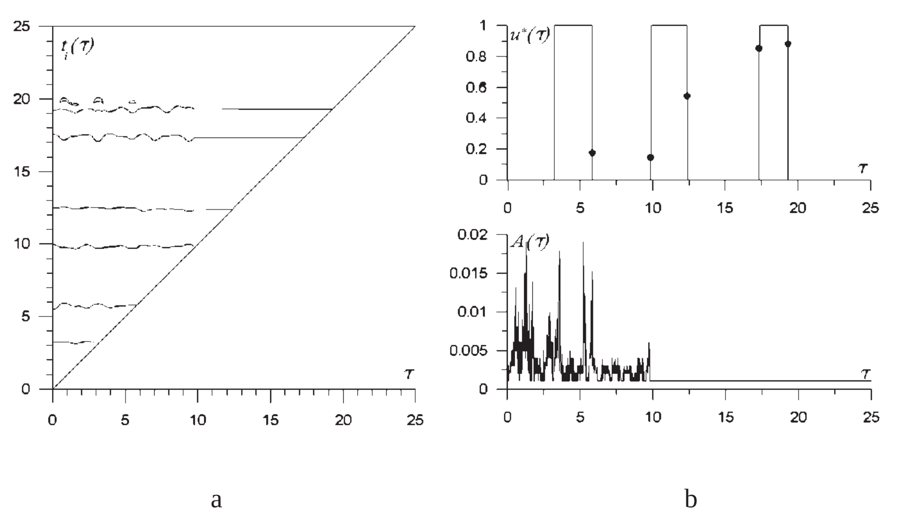

Fig. 4

The set $K_{on} = \{I_{on}, T_{on}\}$, $I_{on} \neq \varnothing$, $T_{on} \neq \varnothing$, is called a support if $\det D_{on} \neq 0$. In the case where $K_{on} = \{I_{on} = \varnothing, T_{on} = \varnothing\}$, the set K_{on} is an empty support by definition.

Main special features of the algorithm of solution of problem (17), as compared with the algorithm of solution of problem (4), are due to the presence of inequality constraints and possible discontinuities of the co-control function at the points $s \in S$. Algorithm of solution of problem (17) is described in [5] together with the algorithm of operation of an optimal controller, based on the dual method of solution of problem (17). As well as in Section 2, the use of an optimal support $K^0_{on}(\tau)$ and the stored auxiliary information allows the correction $K^0_{on}(\tau) \to K^0_{on}(\tau + h)$ to be fulfilled very fast and the control signal $u^*(\tau + h) = u^0(\tau + h | \tau + h, x^*(\tau + h))$ to be calculated.

4. AN OPTIMAL CONTROLLER FOR PIECEWISE LINEAR CONTROL SYSTEMS

Problem Statement. Within the class of discrete controls, let us consider problem (3) of OC of a piecewise linear system.

Discrete control $u(\cdot) = (u(t), t \in T)$ is called admissible control of problem (3) if it satisfies the constraint $|u(t)| \leq 1$, $t \in T$, and the respective path $x(t)$, $t \in T$, the terminal constraint $Hx(t^*) = g$ and intersects boundaries of domains at discrete moments from the set T_h. We will call the admissible control $u^0(\cdot)$ the optimal programmed control of problem (3) if the respective path $x^0(s)$, $t \in T$, provides maximum value to performance criterion of problem (3).

Optimal feedback in problem (3) is determined by expression (8), where X_τ is the set of initial states z for which program OC $u^0(t | \tau, z)$, $t \in T^\tau$, exists in the problem

$$c' x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + bu, \quad x(\tau) = z, \tag{18}$$

$$Hx(t^*) = g, \quad |u(t)| \leq 1, \quad t \in T^\tau,$$

for a fixed $\tau \in T$.

Let us present a scheme of construction of optimal programmed and positional controls in problem (3). Details of the algorithm can be found in [6].

Parametrized Form of the Problem. Each admissible control $u(t)$, $t \in T$, of problem (3) generates a path which sequentially passes through an (admissible) collection $X_{i_1}, \dots, X_{i_k}$, of sets from the population $X_1, \dots, X_p$, by passing through boundaries of adjacent sets at the (admissible) moments $\theta_{i_1}, \dots, \theta_{i_k - 1} \in T_h$. We will designate by $X_1^0, \dots, X_{j^*}^0$ and $\theta_1^0, \dots, \theta_{j^*-1}^0$ a collection of sets and the moments of intersection of boundaries of the sets, which correspond to an optimal path $x^0(t)$, $t \in T$. We will call the sequence of sets $X^0 = \{X_1^0, \dots, X_{j^*}^0\}$ a structure of an optimal path of problem (3). We will assume that the optimal path $x^0(t)$, $t \in T$, intersects boundaries of the sets X_j^0, X_{j+1}^0, $j = \overline{1, j^* - 1}$, along $(n-1)$-dimensional hyperplanes $H_j x = g_j$, where H_j is an $(1 \times n)$-matrix, $g_j \in R$, $j = \overline{1, j^* - 1}$.

The function $\hat{f}(x)$ on the set X_j^0 has the form $\hat{f}(x) = A_j x + a_j$, where $A_j \in R^{n \times n}$, $a_j \in R^n$. Designate $m_j = 1$, $j = \overline{1, j^* - 1}$; $m_{j^*} = \hat{m}$; $H_{j^*} = H$, $g_{j^*} = g$ and put $m = \sum_{j=1}^{j^*} m_j$.

In many problems of the form (3) a set $X_1^0, \dots, X_j^0$ can be generated on the basis of a priori information before solving the problem. Then this problem can be written as an OC problem for a population of linear systems:

$$J(\theta, u) = c' x(\theta_{j^*}) \to \max, \tag{19}$$

$$\dot{x}(t) = A_j x(t) + a_j + bu(t), \quad t \in [\theta_{j-1} \theta_j[, \tag{20}$$

$$j \in J = \{1, 2, \dots, j^*\}, \quad x(\theta_0) = x_0,$$

$$H_j x(\theta_j) = g_i, \quad j \in J, \tag{21}$$

$$|u(t)| \le 1, \quad t \in T, \quad \theta_0 < \ldots < \theta_{j^*-1} < \theta_{j^*} \tag{22}$$

$$\Bigl(x \in R^n, \quad u \in R, \quad g_j \in R^{m_j}, \quad j = \overline{1, j^*}, \quad \operatorname{rank} H_j = m_j < n,$$

$$\theta_j \in T_{h,} \quad j = \overline{1, j^* - 1}, \quad \theta_0 = 0, \quad \theta_{j^*} = t^* \Bigr).$$

The moments $\theta = (\theta_1, \dots, \theta_{j^*-1})$ are selected in problem (19)–(22) alongside with the control $u(t)$, $t \in T$.

We will call the vector θ and discrete control $u(\cdot)$ the admissible control of the problem (19)–(22) if they satisfy constraints (22). We will call the admissible control $\{\theta, u(\cdot)\}$ and the appropriate path $x(t) = x(t | \theta, u(\cdot))$, $t \in T$, of system (20) admissible if $x(t)$, $t \in T$, satisfies constraints (21). The admissible control $\{\theta^0, u^0(\cdot)\}$ is called optimal if performance criterion (19) attains maximum on it.

Problem (19)–(22) can be solved using two procedures: 1) problem (19)–(22) is linearized along an admissible path and solution of the linearized problem is constructed by the method from [4], 2) the solution of the linearized problem is adjusted through the choice of an optimal position of the moments of passage between domains of linearity of the function $\hat{f}(x)$, $x \in X$.

Solution of the Linearized Problem. Linearization of problem (19)–(22) consists in fixing a vector θ corresponding to an admissible path. The linearized problem has the same form (19)–(22) but the vector θ is considered fixed now. Thus, a linear OC problem with intermediate phase constraints is obtained:

$$c' x(t^*) \to \max, \quad \dot{x} = A(t)x + a(t) + bu, \quad x(\theta_0) = x_0,$$

$$H(s)x(s) = g(s), \quad s \in S = \{\theta_1, \dots, \theta_{j^*}\}, \quad |u(t)| \le 1, \quad t \in T. \tag{23}$$

Here $A(t) = A_j$, $a(t) = a_j$, $t \in T_j = [\theta_{j-1}, \theta_j[$, $j = \overline{1, j^*}$; $H(s) = H_j$, $g(s) = g_j$, $s = \theta_j$, and $j = \overline{1, j^*}$.

For solution of problem (23), modification [6] of the algorithm from Section 3 is developed, in which a special structure of the matrix $A(t)$, $t \in T$, is used, and the auxiliary information is presented in the form convenient for the further correction of the moments $\theta_1, \dots, \theta_{j^*-1}$.

Optimization of Transition Moments. As a result of solution of the linearized problem, we obtain an optimal support and a corresponding best value $J^0(\theta)$ of the performance criterion for a fixed vector θ. The purpose is to find a new vector $\bar{\theta}$ for which $J^0(\bar{\theta}) \geq J^0(\theta)$.

Methods of construction of an optimal vector θ^0 [14, 15] are based on calculation of a gradient of performance criterion of problem (3) using θ. It is easy to show that

$$\frac{\partial J(\theta, u)}{\partial \theta_k} = c' \frac{\partial x(t^* | \theta, u^0_{on}(\theta))}{\partial \theta_k} - \nu'(\theta) \left(\begin{matrix} H_j \, \partial x(\theta_j | \theta, u^0_{on}(\theta)) / \partial \theta_k \\ j \in J \end{matrix} \right),$$

$$k = \overline{1, j^* - 1},$$

where $u^0_{on}(\theta) = (u^0(t|\theta),\ \ t \in T^0_{on}(\theta))$, are support values of the OC $u^0(t|\theta)$, $t \in T_h$; $x(t|\theta, u^0_{on}(\theta))$, $t \in T$, is a respective path of system (3). Then a step [14] along the direction grad $J = \left(\frac{\partial J(\theta, u)}{\partial \theta_k},\ \ k = \overline{1, j^* - 1} \right)$ can be calculated and a new vector $\bar{\theta}$ can be constructed.

Optimal Controller. Like above, it is assumed that optimal controller knows the realized state $x^*(\tau)$ at each current instant $\tau \in T_h$ and should work out a signal $u^*(\tau) = u^0(\tau | \tau, x^*(\tau))$.

For this purpose, the controller should solve problem (18) for the position $(\tau, x^*(\tau))$. Formulating problem (18) in a parametric form, we assume that the system dynamics may change not only when the path intersects the hyperplanes $H_j x = g_j$, $j = \overline{1, j^* - 1}$ but also when it is in some ε-neighborhood of these hyperplanes. Inaccuracy due to this assumption can be interpreted as an element of disturbance $w(t)$, $t \in T$. Thus, the parametric form of problem (18) has the form

$$\begin{gathered} c' x(\theta_{j^*}) \to \max, \quad \dot{x}(t) = A_j x(t) + a_j + bu(t), \quad t \in [\theta_{j-1}, \theta_j[, \\ j \in J(\tau) = \{j(\tau), \dots, j^*\}, \quad x(\theta_{j(\tau)-1}) = x^*(\tau), \\ |H_j x(\theta_j) - g_i| \leq \varepsilon, \quad j \in J(\tau) \setminus j^*, \quad Hx(\theta_{j^*}) = g, \quad |u(t)| \leq 1, \quad t \in T^\tau, \\ \theta_{j(\tau)-1} < \theta_{j(\tau)} < \dots < \theta_{j^*-1} < \theta_{j^*}. \end{gathered} \tag{24}$$

The signal $u^*(\tau)$ is calculated based on the optimal support $K^0_{on}(\tau | \theta^0(\tau))$, which can be constructed by the method being proposed [6]. The optimal support $K^0_{on}(\tau - h | \theta^0(\tau - h))$ constructed at the previous moment $\tau - h$ is used as an initial support $K_{on}(\tau | \theta(\tau))$.

5. ASYMPTOTIC CORRECTION OF OPTIMAL CONTROL OF THE APPROXIMATING PROBLEM

As was mentioned in Introduction, the second procedure in solving nonlinear problem (1) consists in asymptotic correction of optimal control of the approximating problem. Optimizations of system (1) in small and in large are distinquished here.

In the former case, system (1) is linearized near a point $x^* \in X$, and the approximating problem is linear. Set (6) represents a quasilinear OC problem

$$c' x(t^*) \to \max, \quad \dot{x} = Ax + a + \mu g(x) + bu, \quad x(0) = x_0,$$

$$x(t^*)\in X^*, \quad |u(t)|\le 1, \quad t\in T, \tag{25}$$

where $A=\partial f(x)/\partial x|_{x=x^*}$, $a=-Ax^*+f(x^*)$.

Kalinin [16, 17] substantiated the asymptotic method of constructing asymptotically s-optimal programmed control for problem (25) for any order of accuracy s. It is based on asymptotic expansion of defining elements of optimal control (the moments of control switching and Lagrange multipliers). This is due to what it differs from the well-known realizations of asymptotic methods based on expansions of direct and dual variables. Such an approach allows one to take into account direct constraints on control, which are serious difficulties for other methods.

In optimization of system (1) in large, the approximating problem is piecewise linear. The method [16, 17] is developed in [7] for piecewise-quasilinear systems. A new point as compared with [16, 17] is the asymptotic expansion of the transition moments $\theta_1,\dots,\theta_{j^*-1}$ between domains of linearity of the function $\hat f(x)$, $x\in X$.

The algorithm of optimal controller operation, realizing a 1-optimal feedback for problems (6), (25) is proposed in [7]. Since optimal controller operates in real time, the algorithm is based on fast recalculation of coefficients of asymptotic expansions, corresponding to a realized state of a nonlinear system. As in Sections 2–4, storage and conversion of the auxiliary information is organized for fast recalculation, which allows one to avoid integration of nonlinear dynamic systems during operation of the controller.

Example 2. Let us illustrate the proposed approach to optimization of nonlinear systems using the problem of optimal damping of a mathematical pendulum

$$\int_0^{10} u(t)\,dt\to\min, \quad \dot x_1=x_2, \quad \dot x_2=-\sin x_1+u, \quad x_1(0)=1.5, \quad x_2(0)=0,$$

$$x_1(10)=x_2(10)=0, \quad 0\le u(t)\le 0.5, \quad t\in T=[0,10], \tag{26}$$

in the domain $X=\{(x_1,x_2):|x_1|<\pi/2\}$.

We will use two approximations of a nonlinear element $\sin x_1$:

1) linear approximation $-x_1$, $x\in X$,

2) piecewise linear approximation $(1-4/\pi)x_1+1-\pi/2$, $x\in X_1=\{(x_1,x_2):\pi/4<x_1<\pi/2\}$; $-x_1$, $x\in X_2=\{(x_1,x_2):|x_1|<\pi/4\}$. Accuracy of the linear approximation is $\delta_1=0.570796$ and of the piecewise linear is $\delta_2=0.110721$.

Thus, in the first case problem (26) is immersed in the set of problems

$$\int_0^{10} u(t)\,dt\to\min, \quad \dot x_1=x_2, \quad \dot x_2=-x_1+\mu(x_1-\sin x_1)/\delta_1+u, \tag{27}$$

$$x_1(0)=1.5, \quad x_2(0)=0, \quad x_1(10)=x_2(10)=0, \quad 0\le u(t)\le 0.5, \quad t\in T,$$

and in the second case in the set
$$\int_0^{10} u(t)\,dt\to\min, \quad \dot x_1=x_2,$$

$$\dot x_2=\begin{cases}(1-4/\pi)x_1+1-\pi/2-\mu((4/\pi-1)x_1-1+\pi/2-\sin x_1)/\delta_2+u, & x\in X_1,\\ -x_1+\mu(x_1-\sin x_1)/\delta_2+u, & x\in X_2,\end{cases} \tag{28}$$

$$x_1(0)=1.5, \quad x_2(0)=0, \quad x_1(10)=x_2(10)=0, \quad 0\le u(t)\le 0.5, \quad t\in T.$$

The parameterized form of problem (28) is

$$\int_0^{10} u(t)\,dt\to\min,$$

TABLE 2

Control	Switching points	Moment θ_1	Performance criterion	Terminal state
$u_1^0(t),\ t\in T$	0.722734, 2.418858 7.005920, 8.702044	—	1.696124	0.017095 –0.516580
$u_1^1(t),\ t\in T$	1.007916, 2.517443 7.547231, 9.017574	—	1.489935	–0.012199 –0.055845
$u_2^0(t),\ t\in T$	1.078442, 2.593439 7.371674, 8.876624	1.233553	1.509974	–0.013850 –0.111839
$u_2^1(t),\ t\in T$	1.064668, 2.573226 7.553250, 9.036841	1.218658	1.496074	–0.001177 –0.008707
$u^0(t,\delta),\ t\in T$	1.064547, 2.573684 7.566068, 9.049386	—	1.496228	10^{-8} 10^{-8}

$$\begin{gathered}\dot{x}_1 = x_2, \quad \dot{x}_2 = (1-4/\pi)x_1 + 1 - \pi/2 \\ + \mu((4/\pi - 1)x_1 - 1 + \pi/2 - \sin x_1)/\delta_2 + u, \quad t\in[0,\theta_1[, \\ x_1(0) = 1.5, \quad x_2(0) = 0, \quad x_1(\theta_1) = \pi/4, \end{gathered} \tag{29}$$

$$\begin{gathered}\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \mu(x_1 - \sin x_1)/\delta_2 + u, \quad t\in[\theta_1, 10], \\ x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \quad t\in T.\end{gathered}$$

Putting in (27), (29) $\mu = 0$, we obtain base problems (16).

Table 2 presents the results of a programmed solution of problem (26). Paths of system (26) were constructed for the following controls:

1) $u_1^0(t)$, $t\in T$, which is optimal control of the linear base problem,

2) $u_1^1(t)$, $t\in T$, which is realization of asymptotically 1-optimal programmed control of problem (27) for the fixed value of $\mu = \delta_1$,

3) $u_2^0(t)$, $t\in T$, which is optimal control of the piecewise linear base problem,

4) $u_2^1(t)$, $t\in T$, which is realization of asymptotically 1-optimal programmed control of problem (29) for the fixed value $\mu = \delta_2$,

5) $u^0(t,\delta)$, $t\in T$, which is optimal programmed control of problem (26) constructed using refining procedure [18]. On each iteration of this procedure the nonlinear system is integrated on the interval T, therefore, it is of interest to compare results in illustrative examples but not for a practical system control in real time.

In each case the control has the form

$$u(t) = \begin{cases} 0, & t\in[0,t_1[\,\cup\,[t_2, t_3[\,\cup\,[t_4, 10[, \\ 0.5, & t\in[t_1,t_2[\,\cup\,[t_3,t_4[\,. \end{cases} \tag{30}$$

Table 2 contains switching points of these controls, values of the moment θ_1 of transition between linearity domains when piecewise linear approximation is used, values of performance criterion, and terminal states of system (26). Curve 1 in Fig. 5 presents the phase path of system (26) constructed under the control $u_2^1(t)$, $t\in T$.

Let us construct a positional solution of problem (26). First, we will analyze behavior of system (26) without disturbances under the action of optimal programmed control $u^0(t,\delta)$, $t\in T$, and under the action of realization $u^{1*}(t)$, $t\in T$, constructed by an 1-optimal controller ($h = 0, 01$). The necessary values of auxiliary functions were calculated by the Runge–Kutta–Felberg method of fourth-fifth order [19]. The control $u^{1*}(t)$, $t\in T$, has the form (30) with switching points

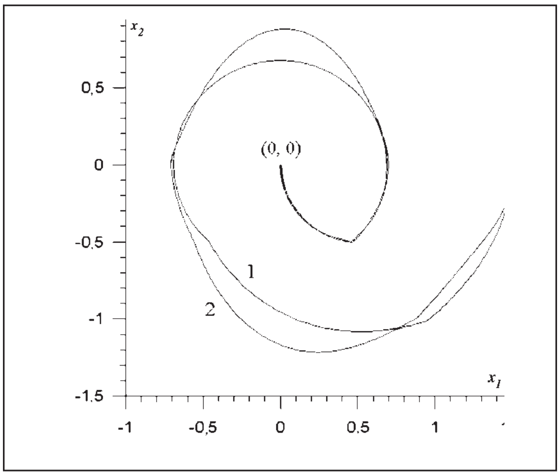

Fig. 5

TABLE 3

Control	Switching points	Performance criterion	Terminal state
$u^{1*}(t),\ t\in T$	1.3, 2.7 7.789350, 9.287743	1.449196	0.000016 –0.001192
$u^{1*}_{10}(t),\ t\in T$	1.3, 2.7 7.717981, 9.257927	1.469973	–0.003723 –0.002823
$u^{1*}_{50}(t),\ t\in T$	1.3, 2.7, 7.778486, 9.278203	1.449858	–0.000074 –0.001229
$u^{1*}_{100}(t),\ t\in T$	1.3, 2.7 7.778420, 9.278330	1.449955	10^{-6} –0.001202
$u^{1*}_{300}(t),\ t\in T$	1.3, 2.7 7.778403, 9.278366	1.449982	0.000014 –0.001133

1.06, 2.58, 7.567630, and 9.039940 and the transition moment between domains of linearity $\theta_1^* = 1.21$. At the moment $t^* = 10$, the path of system (26) under the control $u^{1*}(t)$, $t\in T$, turned out to be in the state (–0.000129, –0.000455), and the value of performance criterion was 1.496155.

Let us consider behavior of the system under disturbance $w^*(t) = 0.4\sin 3t$, $t\in[0, 7[$, $w^*(t)\equiv 0$, $t\ge 7$, unknown to the controller:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\sin x_1 + u + w^*(t),\ x_1(0) = 1.5,\ x_2(0) = 0. \tag{31}$$

Paths of system (31) are constructed for various realizations of optimal feed-back:

1) $u^{1*}(t)$, $t\in T$, constructed by the 1-optimal controller with calculation of necessary values of the functions $x_1(t)$ and $\psi_1(t)$, $t\in T^{\tau}$, $\tau\in T_h$, by the method from [19];

2) $u^{1*}_N(t)$, $t\in T$, constructed by the 1-optimal controller using for the calculations the values of auxiliary functions of the quadrature formula of average rectangles with N nodes for $N = 10, 50, 100, 300$. The constructed controls have the form (30). The realized value of the transition moment is $\theta_1^* = 1.39$ for each of them.

Table 3 contains switching points of the constructed realizations, appropriate values of performance criterion, and terminal states of system (31). Curve 2 in Fig. 5 presents the phase path of system (31) constructed under control $u^{1*}(t)$, $t\in T$.

The results presented in Table 3 show that a rather small number of nodes can be used for calculation using the quadrature formula, i.e., the use of asymptotic correction does not cause an excessive increase in requirements for the size of RAM.

6. SYNTHESIS OF OPTIMAL FEEDFORWARD AND COMBINED COUPLINGS

It was assumed above that there is no possibility to measure disturbances in the process of the control, and current values of control were derived from information on current states. Sometimes (though not often) disturbances can be measured in practice. Adequate use of such information may increase efficiency of control. Closed control systems based on measured disturbances are called feedforward systems (systems of compensating control). If current values of control are derived from current values of state and disturbance, the control system is called closed system with combined relation.

In papers [8–10], the described technique of realization of optimal feedback is transferred to synthesis of optimal feedforward and combined couplings.

CONCLUSIONS

Based of the presented results, we may conclude that the problem of optimal system synthesis posed by engineers over 50 years ago, when computer facilities took first steps, can be solved in practice today if achievements of computing engineering will be used together with modern mathematical methods.

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