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# Optimal Observation of Nonstationary Dynamical Systems

**R. Gabasov, N. M. Dmitruk, and F. M. Kirillova**

*Belarussian State University, ul. Leningradskaya 14, Minsk, 220080 Belarus*
*Institute of Mathematics, Belarussian Academy of Sciences, Belarus*
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**Abstract**—The problem of optimal observation of linear systems is considered. Problems of posterior and positional estimation of the initial and current states of the observed systems by the results of incomplete and inaccurate measurements are solved. The methods suggested are based on a specific (dynamical) implementation of methods of linear programming. The efficiency of the suggested algorithms for optimal observation is illustrated with an example of a fourth-order dynamical system.

## INTRODUCTION

From the early 1960s, when R. Kalman [1] introduced the notions of controllability and observability of dynamical systems, observation problems have been treated as dual to control problems. According to this viewpoint, along with linear problems of optimal control [2] well known at that time, linear problems of optimal observations must be considered. It is clear that such formal reasoning cannot serve as a serious basis for posing and investigating problems of optimal observation. The true basis is the role of problems of optimal observation in the optimization of control systems under conditions of uncertainty, where the available information about the current states of the systems optimized is incomplete and inaccurate. In the case of stochastic uncertainty, analogs of the theory of optimal observation are filter theory, the theory of probabilistic estimation, and the theory of prediction [3]. The main problems in these theories may be considered as dual to problems of stochastic optimal control. Nonstochastic (multiple) uncertainties in the theory of guaranteed optimal control [4] began being exploited only after the modern theory of optimal control of deterministic systems was formed. Therefore, the dual analogs of problems of optimal control did not appear immediately. The first propositions can be found in [5]. Much attention is given to observation problems in [6–8]. In papers [9, 10], problems of optimal observation began to be treated as deterministic analogs of filtering problems [3]. In this approach, linear problems of optimal observation reduced to semiinfinite problems of linear programming. Algorithms for the posterior and positional solution of problems of optimal observation are described in [11]. However, those algorithms do not completely account for the dynamical specificity of the problems under consideration. The purpose of this paper is to realize the approach of [9] with maximal regard for the nature of the problems of optimal observation of dynamical systems and for the conditions of the process of actual solution.

Linear problems of optimal observation, as well as the dual linear problems of optimal control, are not so simple as to expect to obtain their solution in an analytical form. In both cases, the involvement of discrete computational devices cannot be evaded. In this connection, in [12] discrete controls were employed for designing algorithms for the program and positional optimization of dynamical control systems. For the same reason, in this paper discrete signals of measuring devices, which are determined at discrete time instants with a small quantification period, are used. Such an approach corresponds to a number of practical situations and allows one to avoid some analytical problems associated with signals defined on the continuous time axis. The employment of discrete time does not obviate computational difficulties; for small quantification periods, they become quite significant. These difficulties cannot be overcome with the help of only the standard methods of linear programming [13, 14]. In this paper, dynamical implementations of the adaptive method of linear programming [14] are suggested that construct both the posterior and positional solutions to problems of optimal observation at a high rate.

## 1. STATEMENT OF THE PROBLEM

Let the behavior of a dynamical system be described on the time interval $T = [t_*, t^*]$ by the equation

$$\dot{x} = A(t)x, \tag{1.1}$$

where $x = x(t) = (x_j(t), j \in J)$ is the $n$-vector of the state of the system at time instant $t$, $J = \{1, 2, \ldots, n\}$, and $A(t)$, $t \in T$, is a piecewise continuous $n \times n$-matrix function.

Assume that the initial state $x(t_*) = x_0$ of system (1.1) is unknown, yet known to belong to a bounded set

$$X_0 = \{x \in R^n: d_* \le x \le d^*\}$$

(the inequalities are treated coordinate-wise).

Call the set $X_0$ the prior distribution of the state $x_0$ of system (1.1), and call its elements (*a priori*) possible states. The set $X_0$ characterizes the initial uncertainty of system (1.1). It will be seen from what follows that the prior distribution can be written in the form $X_0 = \{x \in R^n: g_* \le Hx \le g^*, d_* \le x \le d^*\}$. This does not influence the efficiency of the methods presented below.

In order to decrease the initial uncertainty of system (1.1), we observe its behavior recording the signals of the measuring device

$$y = c'(t)x + \xi. \quad (1.2)$$

The measuring device (1.2) and its signal $y$ will be called discrete with a quantification period $h > 0$ ($h = (t^* - t_*)/N$, $N$ is a positive integer) if (1.2) takes measurements only at the instants $t \in T_h = \{t_*, t_* + h, \dots, t^*\}$, providing additional information about the unknown realized initial state $x_0$ of system (1.1) in the form of the numbers

$$y(t_*), y(t_* + h), \dots, y(t^*). \quad (1.3)$$

As for the measurement error function $\xi(t)$, $t \in T_h$, we assume it to be sufficiently regular; i.e., it is obtained by the values of a piecewise continuous function $\xi(t)$, $t \in T$, which satisfies the constraints

$$\xi_* \le \xi(t) \le \xi^*, \quad t \in T_h,$$

and has few discontinuity points and bounded derivatives on the continuity intervals.[1] The observation problem consists in processing information (1.3) in order to decrease the prior uncertainty.

**Definition.** The set $\hat{X} = \hat{X}(y(\cdot))$ is called the posterior distribution of the initial state $x_0$ of system (1.1) corresponding to the measurement signal $y(\cdot) = (y(t), t \in T_h)$ if it consists of those and only those vectors $x \in X_0$ that can generate the recorded signals (1.3) together with some possible measurement errors $\xi(t)$, $t \in T_h$.

Under various other names, the set $\hat{X}$ was employed in [5–7]. The choice of denomination for the set $\hat{X}$ is connected with the interpretation of the observation theory as a deterministic analog of stochastic filter theory, in the framework of which prior and posterior probabilistic distributions are widely employed.

[1] In the case where the signal of the measuring device contains both the regular component of the error function $\xi(t)$, $t \in T_h$, and the (noise) irregular (chaotically and rapidly varying) one, the signal is preliminarily filtered, which gives a signal that corresponds to the regular component of the error function.

Bearing in mind the duality between the problems of control and observation, we can treat the set $\hat{X}$ as an analog of the reachability set $X^*$ in the linear problem of optimal control

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0,$$
$$g_* \le Hx(t^*) \le g^*, \quad |u(t)| \le 1, \quad t \in T, \quad (1.4)$$

if we define the set $X^*$ in the following way:

$$X^* = \{x \in R^n: x = x(t^*; x_0, u(\cdot)), |u(t)| \le 1,$$
$$t \in T\} \cap \{x \in R^n: g_* \le Hx \le g^*\}.$$

Here, $x(t^*; x_0, u(\cdot))$ is the state of control system (1.4) at the instant $t^*$ corresponding to the initial state $x(t_*) = x_0$ and control $u(\cdot) = (u(t), t \in T)$.

The reachability set characterizes the possibilities of control system (1.4); the posterior distribution reflects the possibilities of observation system (1.1), (1.2).

In general (where the number of measurements $(t^* - t_*)/N$ is large), the set $\hat{X}$ has a very complex structure; it can hardly be constructively described and employed for solving problems of guaranteed optimal control [11]. The situation with the reachability set $X^*$ is quite similar. Fortunately, when solving problems of guaranteed optimal control, one employs only linear estimates for the set $\hat{X}$ of the form

$$\hat{\alpha} = \max_{x \in \hat{X}} p'x, \quad (1.5)$$

where $p$ is a given $n$-vector, $\|p\| = 1$. Geometrically, the estimate $\hat{\alpha}$ is equal to the maximal projection of the set $\hat{X}$ along $p$.

We call elements $x$ of the set $\hat{X}$ (posteriorly) possible (initial) states of system (1.1). A possible state $x^*$ is called extremal ($p$-extremal or extremal for the direction $p$) if

$$p'x^* = \max_{x \in \hat{X}} p'x.$$

A possible state $x^\varepsilon$ is $\varepsilon$-extremal for a given $\varepsilon \ge 0$ if

$$p'x^* - p'x^\varepsilon \le \varepsilon.$$

The posterior solution of the problem of optimal observation we refer to as the construction of an extremal state and the corresponding estimate. The positional solution is defined in Section 6.

Extremal problem (1.5) is called the linear problem of optimal observation.[2] The purpose of the constructions below is the efficient solution of this problem.

[2] The problem of optimal observation investigated in [6] differs from the presented one both in essence and in methods of solution.

Knowing the posterior distribution $\hat{X}$ of the state $x_0$ of system (1.1), one can find the posterior distribution $\hat{X}(\tau)$ of the current state $x(\tau)$

$$\hat{X}(\tau) = F(\tau, t_*)\hat{X} = \{x \in R^n: x = F(\tau, t_*)x_0, x_0 \in \hat{X}\}.$$

Here, $F(t, t_*) = F(t)F^{-1}(t_*)$, $F(t)$, $t \in T$, is the fundamental $n \times n$-matrix of solutions to equation (1.1): $\dot{F} = A(t)F$, $F(0) = E$, $E$ is the identity diagonal $n \times n$-matrix.

To give an analytical formulation of problem (1.5), let us first analytically describe the set $\hat{X}$. For the initial state $x \in X_0$ and the error function $\xi(t)$, $t \in T_h$, the signal of measuring device (1.2) has the form

$$\begin{aligned} y(t) &= c'(t)x(t) + \xi(t) \\ &= c'(t)F(t, t_*)x + \xi(t), \quad t \in T_h. \end{aligned}$$

Thus, for a measured signal $y(t)$, $t \in T_h$, the set $\hat{X}$ consists of those and only those points $x \in R^n$ that satisfy the relations

$$\xi_* \le y(t) - c'(t)F(t, t_*)x \le \xi^*,$$
$$t \in T_h; \quad x \in X_0.$$

Introduce the notation $\xi_*(t) = \xi_* - y(t)$, $\xi^*(t) = \xi^* - y(t)$, $d'(t) = -c'(t)F(t, t_*)$, $t \in T_h$. Then, the problem of optimal observation (1.5) takes the form

$$\begin{gathered} p'x \longrightarrow \max, \\ \xi_*(t) \le d'(t)x \le \xi^*(t), \quad t \in T_h, \\ d_* \le x \le d^*. \end{gathered} \tag{1.6}$$

The interval problem of linear programming (1.6) has the size $(N + 1) \times n$; i.e., for small quantification periods $h$, it is semilarge in the sense that it has a large number of main constraints for a fixed number $n$ of variables. In this respect, it is dual to the semilarge problem of optimal control (1.4) in the class of discrete controls ($u(t) = U(t_* + kh)$; $t \in [t_* + kh, t_* + (k + 1)h)$, $k = \overline{1, N-1}$), where the number of variables for a fixed number of main constraints grows with decreasing quantification period.

The standard methods of linear programming [13, 14] are not efficient for solving semilarge linear problems. Therefore, we direct our further efforts toward generalizing the adaptive method of linear programming [14] in order to efficiently solve semilarge problem (1.6). The main tool of the adaptive method is the support. In this connection, we start the construction by introducing its dynamical analog.

## 2. THE SUPPORT OF THE PROBLEM OF OPTIMAL OBSERVATION

In the sets $T_h$ and $J$, let us choose arbitrary subsets $T_{sup} \subset T_h$ and $J_{sup} \subset J$ having the same number of elements: $|T_{sup}| = |J_{sup}|$. Compose the following matrix:

$$D_{sup} = \begin{pmatrix} d_j(t), & (j \in J_{sup}) \\ t \in T_{sup} & \end{pmatrix}.$$

Call the pair $K_{sup} = \{T_{sup}, J_{sup}\}$ a support of problem (1.6) if $\det D_{sup} \neq 0$. In the case of empty sets $T_{sup} = \varnothing$, $J_{sup} = \varnothing$, the pair $K_{sup}$ is an (empty) support by definition.

Let us present two "dynamical" ways of constructing the support matrix $D_{sup}$.

**The direct method.** For each $j \in J_{sup}$, integrate direct system (1.1) with the initial condition $x(t_*) = e_j$ ($e_j$ is the $j$th unit vector) and, on the trajectory $x^{(j)}(t)$, $t \in T$, calculate the values of the output signal $z^{(j)}(t) = -c'(t)x^{(j)}(t)$ at the support instants $t \in T_{sup}$. Then,

$$D_{sup} = \begin{pmatrix} z^{(j)}(t), & j \in J_{sup} \\ t \in T_{sup} & \end{pmatrix}.$$

**The dual method.** Let $\psi^{(l)}(t_*)$ be the costate of the dual system

$$\dot{\psi} = -A'(t)\psi \tag{2.1}$$

with the initial condition $\psi(t_l) = -c(t_l)$, $t_l \in T_{sup}$, at the instant $t_*$. Then,

$$D_{sup} = \begin{pmatrix} \psi_j^{(l)}(t_*), & j \in J_{sup} \\ t_l \in T_{sup} & \end{pmatrix}.$$

In the method presented below, along with the support, we use the companion elements: (1) the function of potentials $\nu(t)$, $t \in T_h$; (2) the vector of estimates $\Delta = \Delta(J) = (\Delta_j, j \in J)$; (3) the pseudostate $\kappa = \kappa(J) = (\kappa_j, j \in J)$; and (4) the output pseudosignal $\zeta(t)$, $t \in T_h$.

The construction of the function of potentials starts with defining its values at nonsupport instants $t \in T_n = T_h \backslash T_{sup}$: $\nu(t) = 0$, $t \in T_n$. The support values $\nu_{sup} = \nu(T_{sup}) = (\nu(t), t \in T_{sup})$ can be found from the equation

$$\nu'_{sup} D_{sup} = p'_{sup}, \tag{2.2}$$

where $p_{sup} = p(J_{sup}) = (p_j, j \in J_{sup})$. In the case of an empty support $K_{sup} = \varnothing$, we assume $\nu(t) = 0$, $t \in T_h$.

The vector of estimates is calculated by the formula

$$\Delta' = p' - \nu'_{sup} D_{(sup)}, \tag{2.3}$$

where $D_{(\text{sup})}$ is the matrix composed of the rows $d(t)$, $t \in T_{\text{sup}}$. Here, $\Delta_{\text{sup}} = (\Delta_j, j \in J_{\text{sup}}) = 0$. The matrix $D_{(\text{sup})}$ is constructed similarly to the matrix $D_{\text{sup}}$ by the direct or dual method. If $K_{\text{sup}} = \varnothing$, then $\Delta = p$.

To construct the pseudostates $\kappa$ and the output pseudosignal $\zeta(t)$, $t \in T_h$, first specify the nonsupport components of the pseudostate $\kappa_{\text{n}} = \kappa(J_{\text{n}})$:

$$\kappa_j = d_{*j}, \text{ if } \Delta_j < 0; \quad \kappa_j = d_j^*, \text{ if } \Delta_j > 0; \quad \kappa_j \in [d_{*j}, d_j^*], \text{ if } \Delta_j = 0; \quad j \in J_{\text{n}}; \tag{2.4}$$

and the support values of the output pseudosignal $\zeta_{\text{sup}} = \zeta(T_{\text{sup}})$

$$\zeta(t) = \xi_*, \text{ if } \nu(t) < 0; \quad \zeta(t) = \xi^*, \text{ if } \nu(t) > 0; \quad \zeta(t) \in [\xi_*, \xi^*], \text{ if } \nu(t) = 0; \quad t \in T_{\text{sup}}. \tag{2.5}$$

The support components of the pseudostate $\kappa_{\text{sup}} = \kappa(J_{\text{sup}}) = (\kappa_j, j \in J_{\text{sup}})$ can be found from the system of equations

$$\sum_{j \in J_{\text{sup}}} d_j(t)\kappa_j = \zeta(t) - y(t) - \sum_{j \in J_{\text{n}}} d_j(t)\kappa_j, \quad t \in T_{\text{sup}}. \tag{2.6}$$

Let us also present a dynamical method for determining the components of the vector $\kappa_{\text{sup}}$. Let $\kappa_0(t)$, $t \in T$, be the trajectory of system (1.1) with the initial condition $x(t_*) = x_0$, where $x_0 = x_0(J) = (x_{0j} = 0, j \in J_{\text{sup}}; x_{0j} = \kappa_j, j \in J_{\text{n}})$; and $\zeta_0(t) = c'(t)\kappa_0(t)$, $t \in T_h$, be the output signal of the observation device. Then, the system of equations (2.6) takes the form

$$D_{\text{sup}}\kappa_{\text{sup}} = (\zeta(t) - y(t) + \zeta_0(t), \, t \in T_{\text{sup}}).$$

Let the function $\kappa(t)$, $t \in T_h$, which is the solution to system (1.1) with the initial condition $x(t_*) = \kappa$, be referred to as the companion pseudotrajectory for a support $K_{\text{sup}}$. Then, the output pseudosignal is $\zeta(t) = y(t) - c'(t)\kappa(t) = y(t) + d'(t)\kappa$, $t \in T_h$.

Let a support $K_{\text{sup}}$ be called regular if its companion function of potentials and the vector of estimates satisfy the relations $\nu(t) \neq 0$, $t \in T_{\text{sup}}$; $\Delta_j \neq 0$, $j \in J_{\text{n}}$. Let a pair $\{x, K_{\text{sup}}\}$ composed of a posteriorly possible state $x$ and a support $K_{\text{sup}}$ be referred to as a support state. A support state is called directly nonsingular if $d_{*j} < x_j < d_j^*$, $j \in J_{\text{sup}}$; $\xi_* < z(t) < \xi^*$, $t \in T_{\text{n}}$ where $z(t) = y(t) - c'(t)x(t) = y(t) + d'(t)x$, $t \in T_h$, is the output signal. It is called dually nonsingular if the support $K_{\text{sup}}$ is regular.

The number

$$\beta(x, K_{\text{sup}}) = p'\kappa - p'x = \sum_{t \in T_{\text{sup}}} \nu(t)(\zeta(t) - z(t)) + \sum_{j \in J_{\text{n}}} \Delta_j(\kappa_j - x_j)$$

is called the suboptimality estimate for the support state $\{x, K_{\text{sup}}\}$.

## 3. CRITERIA OF OPTIMALITY AND SUBOPTIMALITY

Let us use the support, first of all, to formulate the main results of the qualitative theory of optimal observation. These results follow from [14].

**The maximum principle.** For the extremality of a posteriorly possible state $x^*$, it is necessary and sufficient that there exists a support $K_{\text{sup}}$ such that the support state $\{x^*, K_{\text{sup}}\}$ and its companion elements satisfy

(1) the maximum condition with respect to the state $\Delta' x^* = \max_{d_* \le x \le d^*} \Delta' x$;

(2) the maximum condition with respect to the output signal $z^*(t) = y(t) - c'(t)x^*(t)$, $t \in T_h$,

$$\nu(t)z^*(t) = \max_{\xi_* \le z \le \xi^*} \nu(t)z, \quad t \in T_{\text{sup}}.$$

The support $K_{\text{sup}}$, the existence of which is mentioned in the maximum principle, is called optimal.

The maximum principle implies as a corollary the following criterion for the optimality of a support, which is employed in the dual method for solving the problem of optimal observation:

For the optimality of a support $K_{\text{sup}}$, it is necessary and sufficient that a certain companion pseudostate $\kappa$ and a certain output pseudosignal $\zeta(t)$, $t \in T_h$, satisfy the relations

$$d_{*j} \le \kappa_j \le d_j^*, \quad j \in J_{\text{sup}}; \quad \xi_* \le \zeta(t) \le \xi^*, \quad t \in T_{\text{n}}. \tag{3.1}$$

**The $\varepsilon$-maximum principle.** Given any $\varepsilon \ge 0$, for the $\varepsilon$-extremality of a posteriorly possible state $x^\varepsilon$, it is necessary and sufficient that there exists a support $K_{\text{sup}}$ such that its companion elements satisfy

(1) the $\varepsilon$-maximum condition with respect to the state

$$\Delta_j x_j^\varepsilon = \max_{d_{*j} \le x_j \le d_j^*} \Delta_j x_j - \varepsilon_{xj}, \quad j \in J_{\text{n}};$$

(2) the $\varepsilon$-maximum condition with respect to the output signal

$$\nu(t)z^\varepsilon(t) = \max_{\xi_* \le z \le \xi^*} \nu(t)z - \varepsilon_z(t), \quad t \in T_{\text{sup}}.$$

(3) the ε-accuracy condition

$$\sum_{t \in T_{sup}} \varepsilon_z(t) + \sum_{j \in J_n} \varepsilon_{xj} \le \varepsilon.$$

## 4. THE DIRECT METHOD

The direct method for constructing a posterior solution to problem (1.5) is iterative. At each iteration, the old support state $\{x, K_{sup}\}$ is replaced with a new one $\{x, \bar{K}_{sup}\}$ such that the suboptimality estimate decreases: $\beta(\bar{x}, \bar{K}_{sup}) \le \beta(x, K_{sup})$. We realize the iteration step of the direct method in the form of two procedures: (1) a change of the possible state $x \longrightarrow \bar{x}$; (2) a change of the support $K_{sup} \longrightarrow \bar{K}_{sup}$. The task of the first phase, which can be accomplished [14] with the help of the method presented below in this section, is the construction of the initial support state.

To simplify the calculations, we assume below that, at the iterations of the direct method, only directly and dually nonsingular support states are used $\{x, K_{sup}\}$ and that, at the iterations of the dual method, only regular supports are employed; the realized error function is continuous.

We refer to an instant $t \in T_h \backslash \{t_*, t^*\}$ as a minimum point of the output signal $z(t)$, $t \in T_h$ if $z(t) < z(t-h)$ and $z(t) < z(t+h)$, or a maximum point if $z(t) > z(t-h)$ and $z(t) > z(t+h)$. Denote by $T_0^-$ the set of minimum points of the output signal, by $T_0^+$ the set of maximum points, $T_0 = T_0^- \cup T_0^+$.

The sets $T_0^-$ and $T_0^+$ are constructed before the first iteration simultaneously with the integration of the direct system (1.1) and the construction of the output signal $z(t)$, $t \in T_h$.

Suppose that at the beginning of the first iteration the following information is known (stored in the computer memory): (1) the possible state $x$, (2) the support $K_{sup}$, (3) the matrix $D_{(sup)}$, (4) the support values of the function of potentials $\nu_{sup}$, (5) the nonsupport components of the vector of estimates $\Delta_n$, (6) the pseudostate $\kappa$, (7) the sets $T_0^+$ and $T_0^-$, (8) the values of the matrix function $F(t)$ at the instants[3] $t \in T_{sup} \cup T_0 \cup \{t_*, t^*\}$, and (9) the suboptimality estimate $\beta(x, K_{sup})$.

One can easily restore the extremal values of the output signal and pseudosignal by data (7) and (8): $z(t) = y(t) - c'(t)F(t)F^{-1}(t_*)x$, $\zeta(t) = y(t) - c'(t)F(t)F^{-1}(t_*)\kappa$, $t \in T_0$.

[3] If the function $\xi(t)$, $t \in T$, has a discontinuity at a point of the closed interval $[t, t+h]$, then the values of the fundamental matrix at the points $t$ and $t+h$ are also stored.

Before starting the iteration, we make sure that on the support state (for a chosen $\varepsilon > 0$) the ε-maximum principle does not hold and that inequalities (3.1) are violated.

**Change of the possible state.** Construct a new possible state $\bar{x}$ by the rule $\bar{x} = x + \theta^0 l$, where $l = \kappa - x$ and $\theta^0$ is the maximal step along $l$ such that the inequalities $d_{*j} \le x_j \le d_j^*$, $j \in J_{sup}$; and $\xi_* \le z(t) \le \xi^*$, $t \in T_n$, are not violated.

To find the direct step $\theta^0$, we determine

$$\theta_j,\ j \in J_{sup};\ \theta(t),\ \tau(t),\ \theta_0(t),\ t \in T_0 \cup \{t_*, t^*\}; \quad (4.1)$$

by the rules

$$\theta_j = \begin{cases} (d_{*j} - x_j)/l_j, & \text{if } l_j < 0 \\ (d_j^* - x_j)/l_j, & \text{if } l_j > 0 \\ +\infty, & \text{if } l_j = 0; \quad j \in J_{sup}; \end{cases}$$

$$\theta(t) = \begin{cases} (\xi_* - z(t))/l_z(t), & \text{if } l_z(t) < 0 \\ (\xi^* - z(t)/l_z(t), & \text{if } l_z(t) > 0 \\ +\infty, & \text{if } l_z(t) = 0; \end{cases}$$

$$l_z(t) = \zeta(t) - z(t), \quad t \in T_0 \cup \{t_*, t^*\};$$

$$\begin{aligned} \tau(t) &= -1, \text{ if } t \in T_0^- \text{ and } \dot{d}'(t)l > 0; \\ &\text{or } t \in T_0^+ \text{ and } \dot{d}'(t)l < 0; \qquad (4.2)\\ \tau(t) &= 1, \text{ if } t \in T_0^+ \text{ and } \dot{d}'(t)l > 0; \\ &\text{or } t \in T_0^- \text{ and } \dot{d}'(t)l < 0; \\ \tau(t_*) &= 1, \quad \tau(t^*) = -1; \end{aligned}$$

$$\theta_0(t) = -\frac{z(t+\tau(t)h) - z(t)}{l_z(t+\tau(t)h) - l_z(t)}$$
$$= -\frac{(c'(t+\tau(t)h)F(t+\tau(t)h) - c'(t)F(t))F^{-1}(t_*)x}{(c'(t+\tau(t)h)F(t+\tau(t)h) - c'(t)F(t))F^{-1}(t_*)l}. \quad (4.3)$$

Here, $\dot{d}'(t) = -\dot{c}'(t)F(t)F^{-1}(t_*) - c'(t)A(t)F(t)F^{-1}(t_*)$.

The number $\theta_j$ is the step along $l$ such that the $j$th direct constraint becomes active; $\theta(t)$ is the step such that the output signal $z(t) + \theta l_z(t)$, $t \in T_h$, attains the boundary $\xi_*$ of $\xi^*$ at an extremum point $t \in T_0$ or at the ends of the interval $T_h$; the numbers $\tau(t)$, $t \in T_0$, indicate a possible direction of motion of the maximum and minimum points; $\theta_0(t) \ge 0$ is the step along $l_z(t)$, $t \in T_h$, such that the extremum point of the output signal moves from the point $t$ into $t + \tau(t)h$. Steps $\theta_0(t) < 0$ are not taken into account.

We calculate the direct step $\theta^0$ by stages. Let the $(i-1)$th stage along $l$ be accomplished and, before the $i$th stage, the sets $T_0^{-i}$ and $T_0^{+i}$ and the steps $\theta^i(t)$ and $\theta_0^i(t)$, $t \in T_0^i = T_0^{-i} \cup T_0^{+i}$, be known.

To simplify the calculations, we assume that all the numbers $\theta^i(t)$, $\theta_0^i(t)$, $t \in T_0^i$, $t \in T_0^i \cup \{t_*, t^*\}$, are different, perhaps with the exception of the numbers $\theta_0^i(t)$, $\theta_0^i(t+h)$, $t, t+h \in T_0^i$, where $\tau(t-h) = 1$, $\tau(t+h) = -1$.

Calculate the step

$$\theta^i = \min\{1, \theta_{j_0}, \theta(t_0^i), \theta_0(t^i)\};$$

$$\theta_{j_0} = \min_{j \in J_{\rm sup}} \theta_j; \quad \theta^i(t_0^i) = \min \theta^i(t), \quad t \in T_0^i \cup \{t_*, t^*\}; \tag{4.4}$$

$$\theta_0^i(t^i) = \min \theta_0^i(t); \quad t \in T_0^i \cup \{t_*, t^*\}.$$

In the case where $\theta^i = \theta_{j_0}$ or $\theta^i = \theta^i(t_0^i)$, we set $\theta^0 = \theta^i$ and $t_0 = t_0^i$, fix the sets $\bar{T}_0^-$ and $\bar{T}_0^+$ for the new iteration: $\bar{T}_0^- = T_0^{-i}$, $\bar{T}_0^+ = T_0^{+i}$, and pass to the procedure of changing the support. For $\theta^i = \theta_0^i(t^i)$, we transform the information stored in the memory in the following way.

(1) Let $\theta^i = \theta_0^i(t^i)$, $t^i \in T_0^i$, be the unique number on which the minimum in (4.4) is attained. Then, instead of the instant $t^i$, store the instant $t^i + \tau(t^i)h$ in $T_0^i$: $T_0^{i+1} = (T_0^i \backslash t^i) \cup \{t^i + \tau(t^i)h\}$ and, instead of the matrix $F(t^i)$, store $F(t^i + \tau(t^i)h)$ after integrating the direct equation $\dot{F} = A(t)F$ on the interval $[t^i, t^i + h]$ or $[t^i - h, t^i]$. Calculate the new steps $\theta^{i+1}(t^i + \tau(t^i)h)$, $\theta_0^{i+1}(t^i + \tau(t^i)h)$ by formulas (4.2) and (4.3).

(2) If $t^i = t_*$ or $t^i = t^*$, then we put the instant $t^i + \tau(t^i)h$ into the set $T_0^i$: $T_0^{i+1} = T_0^i \cup \{t^i + \tau(t^i)h\}$. Calculate and store $F(t^i + \tau(t^i)h)$; set $\tau(t^i + \tau(t^i)h) = \tau(t^i)$. Treat the situation as the appearance of a new extremum point at the end of the interval $T_h$. Calculate the steps $\theta^{i+1}(t^i + \tau(t^i)h)\theta_0^{i+1}(t^i + \tau(t^i)h)$. Comparing the values $z(t^i)$ and $z(t^i + \tau(t^i)h)$, find out whether $t^i + \tau(t^i)h$ is the minimum or maximum point of the output signal $z(t) + \theta l_z(t)$, $t^i + \tau(t^i)h$ is the minimum point if $z(t^i) < z(t^i + \tau(t^i)h)$, or the maximum point if $z(t^i) > z(t^i + \tau(t^i)h)$.

(3) For $\theta^i = \theta_0^i(t^i) = \theta_0^i(t^i + h)$, $\tau(t^i) = 1$, $\tau(t^i + h) = -1$, $t^i, t^i + h \in T_0$, remove the points $t^i$ and $t^i + h$ from the set $T_0^i$: $T_0^{i+1} = T_0^i \backslash \{t^i, t^i + h\}$, since, for the step $\theta > \theta^i$, the points $t^i$ and $t^i + h$ are no longer the extremum points of the output signal. Remove the values $F(t^i)$ and $F(t^i + h)$ of the fundamental matrix from the computer memory.

(4) If $\theta^i = \theta_0^i(t^i) = \theta_0^i(t^i + h)$, $\tau(t^i) = 1$, $\tau(t^i + h) = -1$, and $t^i = t_*$, then treat the situation as the disappearance of an extremum point through the left end of the interval $T_h$. Remove the point $t^i + h$ from the set $T_0^i$: $T_0^{i+1} = T_0^i \backslash t^i + h$; remove the value $F(t^i + h)$ of the fundamental matrix from the computer memory.

(5) For $\theta^i = \theta_0^i(t^i) = \theta_0^i(t^i + h)$, $\tau(t^i) = 1$, $\theta_0^i(t^i + h) = -1$, $t^i + h = t^*$, the extremum point disappears through the right end of the interval $T_h$. The new set of extremal points has the form $T_0^{i+1} = T_0^i \backslash t^i$. Delete the value $F(t^i)$ from the memory.

The suboptimality estimate for the support state $\{\bar{x}, K_{\rm sup}\}$ is $\beta(\bar{x}, K_{\rm sup}) = (1 - \theta^0)\beta(x, K_{\rm sup})$. For $\beta(\bar{x}, \bar{K}_{\rm sup}) \le \varepsilon$, the process of solution of problem (1.6) is terminated, because $\bar{x}$ is an $\varepsilon$-extremal state. Otherwise, pass to the procedure of changing the support.

**Change of the support.** The rules of the support change depend on two possible (after the first procedure) situations: (1) $\theta^0 = \theta_{j_0}$; (2) $\theta^0 = \theta(t_0)$. Let us investigate these situations individually.

(1) Let $\theta^0 = \theta_{j_0}$. Following [14], construct the variations of the function of potentials $\Delta\nu(t)$, $t \in T_h$, and the vector of estimates $\Delta\delta$. For this purpose, first, specify the values of the variation of the function of potentials at the nonsupport instants: $\Delta\nu(t) = 0$, $t \in T_{\rm n}$, and the variations of the support components of the vector of estimates $\Delta\delta_{\rm sup} = \Delta\delta(J_{\rm sup})$: $\Delta\delta_{j_0} = 1$ if $\kappa_{j_0} > d^*_{j_0}$; $\Delta\delta_{j_0} = -1$ if $\kappa_{j_0} < d_{*j_0}$, and $\Delta\delta_j = 0$, $j \in J_{\rm sup} \backslash j_0$. Find the values $\Delta\nu_{\rm sup} = \Delta\nu(T_{\rm sup}) = (\Delta\nu(t), t \in T_{\rm sup})$ from the equation

$$\Delta\nu'_{\rm sup} D_{\rm sup} = -\Delta\delta'_{\rm sup}.$$

The function $\Delta\nu(t)$, $t \in T_h$, generates the variation of the vector of estimates

$$\Delta\delta' = -\Delta\nu'_{\rm sup} D_{(\rm sup)} = -\sum_{t \in T_{\rm sup}} \Delta\nu(t) d'(t).$$

(2) For $\theta^0 = \theta(t_0)$, construct the variation of the function of potentials $\Delta\nu(t)$, $t \in T_h$, by the following rules: $\Delta\nu(t_0) = 1$ if $\zeta(t_0) > \xi^*$, $\Delta\nu(t_0) = -1$ if $\zeta(t_0) < \xi_*$, and $\Delta\nu(t) = 0$, $t \in T_{\rm n} \backslash t_0$; find the values $\Delta\nu_{\rm sup}$ from the equation

$$\Delta\nu'_{\rm sup} D_{\rm sup} = -d'_{\rm sup}(t_0)\Delta\nu(t_0),$$

where $d_{\rm sup}(t_0) = (d_j(t_0), j \in J_{\rm sup})$.

Find the variation of the vector of estimates by the formula

$$\Delta\delta' = -\Delta\nu'_{\text{sup}}D_{(\text{sup})} - \Delta\nu(t_0)d'(t_0).$$

It is known [14] that the initial rate of change of the quality criterion of the problem dual to (1.6) along $\Delta\nu(t)$, $t \in T_h$; $\Delta\delta$, is

$$\alpha^1 = \begin{cases} -\rho(\kappa_{j_0}, [d_{*j_0}, d^*_{j_0}]), \text{ if } \theta^0 = \theta_{j_0} \\ -\rho(\zeta(t_0), [\xi_*, \xi^*]), \text{ if } \theta^0 = \theta(t_0) \end{cases} \quad \alpha^1 < 0,$$

where $\rho(c, [a, b])$ is the distance between the number $c$ and the closed interval $[a, b]$.

Calculate $n$ numbers

$$\sigma_j = \begin{cases} -\Delta_j/\Delta\delta_j, \text{ if } \Delta_j\Delta\delta_j < 0 \\ \infty, \text{ if } \Delta_j\Delta\delta_j \ge 0; \quad j \in J_n; \end{cases}$$

$$\sigma(t) = \begin{cases} -\nu(t)/\Delta\nu(t), \text{ if } \nu(t)\Delta\nu(t) < 0 \\ \infty, \text{ if } \nu(t)\Delta\nu(t) \ge 0; \quad t \in T_{\text{sup}}. \end{cases}$$

The number $\sigma_j$ represents the step along $\Delta\delta$ such that the $j$th component of the disturbed vector of estimates $\delta(\sigma) = \Delta + \sigma\Delta\delta$ vanishes. Similarly, making a step $\sigma(t)$ along $\Delta\nu(t)$, $t \in T_h$, we obtain the zero value of the disturbed function of potentials $\nu(\sigma, t) = \nu(t) + \sigma\Delta\nu(t)$, $t \in T_h$ at the instant $t \in T_{\text{sup}}$.

To simplify the calculations, we assume that all the numbers $\sigma_j$, $j \in J_n$, and $\sigma(t)$, $t \in T_{\text{sup}}$, are different and nonzero. Enumerate them in the ascending order:

$$0 < \sigma^1 < \sigma^2 < \ldots < \sigma^n.$$

Let the rate of decrease of the dual quality criterion on the interval $[\sigma^{k-1}, \sigma^k[$, $k \ge 1$, $(\sigma^0 = 0)$ be $\alpha^k < 0$. At the point $\sigma^k$, this rate has a discontinuity

$$\Delta\alpha^k = \begin{cases} (d^*_{j^k} - d_{*j^k})|\Delta\delta_{j^k}|, \text{ if } \sigma^k = \sigma_{j^k} \\ (\xi^* - \xi_*)|\Delta\nu(t^k)|, \text{ if } \sigma^k = \sigma(t^k), \end{cases}$$

and, at the interval $[\sigma^k, \sigma^{k+1}[$ it is equal to $\alpha^{k+1} = \alpha^k + \Delta\alpha^k$. The number $\sigma^* = \sigma^{k_*}$ such that $\alpha^{k_*} < 0$, $\sigma^{k_*+1} \ge 0$ is called a long dual step.

We shall distinguish two cases: (a) $\sigma^* = \sigma_{j^{k_*}}$ and (b) $\sigma^* = \sigma(t^{k_*})$.

The new support $\bar{K}_{\text{sup}} = \{\bar{T}_{\text{sup}}, \bar{J}_{\text{sup}}\}$ is constructed by the following rules: (1a) $\bar{T}_{\text{sup}} = T_{\text{sup}}$, $\bar{J}_{\text{sup}} = (J_{\text{sup}}\backslash j_0) \cup j^{k_*}$; (1b) $\bar{T}_{\text{sup}} = T_{\text{sup}}\backslash t^{k_*}$, $\bar{J}_{\text{sup}} = J_{\text{sup}}\backslash j_0$; (2a) $\bar{T}_{\text{sup}} = T_{\text{sup}} \cup t_0$, $\bar{J}_{\text{sup}} = J_{\text{sup}} \cup j^{k_*}$; (2b) $\bar{T}_{\text{sup}} = (T_{\text{sup}}\backslash t^{k_*}) \cup t_0$, $\bar{J}_{\text{sup}} = J_{\text{sup}}$.

The suboptimality estimate for the new support state $\{\bar{x}, \bar{K}_{\text{sup}}\}$ is

$$\beta(\bar{x}, \bar{K}_{\text{sup}}) = (1-\theta^0)\beta(x, K_{\text{sup}}) + \sum_{k=1}^{k_*}\alpha^k(\sigma^k - \sigma^{k-1}).$$

If $\beta(\bar{x}, \bar{K}_{\text{sup}}) \le \varepsilon$, then the process of solution $x$ of problem (1.6) is terminated. Otherwise, transform data (3)–(6).

In case (1a), we retain the matrix $D_{(\text{sup})}$ unchanged: $\bar{D}_{(\text{sup})} = D_{(\text{sup})}$; in case (1b), we obtain $\bar{D}_{(\text{sup})}$ by deleting the row corresponding to the instant $t^{k_*} \in T_{\text{sup}}$ from $D_{(\text{sup})}$. To construct the support matrix $\bar{D}_{\text{sup}}$, distinguish in $\bar{D}_{(\text{sup})}$ the columns with the indices $j \in \bar{J}_{\text{sup}}$. In case (2), to obtain $\bar{D}_{(\text{sup})}$, we add the row $d(t_0)$ to the matrix $D_{(\text{sup})}$; in case (2b), we additionally delete the row $d(t^{k_*})$.

The support values of the function of potentials $\bar{\nu}_{\text{sup}}$ and the vector of estimates are recalculated by formulas (2.2) and (2.3).

Construct the nonsupport components of the pseudostate $\bar{\kappa}_n = \bar{\kappa}(\bar{J}_n)$ and the values of the output pseudosignal $\bar{\zeta}(t)$ at the support instants $t \in \bar{T}_{\text{sup}}$ by rules (2.4) and (2.5). Solving system (2.6), we find $\bar{\kappa}_{\text{sup}} = \bar{\kappa}(\bar{T}_{\text{sup}})$.

**Theorem 1.** The method is finite if only directly and dually nonsingular support states are used at its iterations.

One can construct a modification of the adaptive method [14], which is finite for any problem (1.6).

## 5. THE DUAL METHOD

The method is intended to construct the posterior solution to the problem of optimal observation (1.5) and is iterative. At its iteration, the old support is replaced with a new one $K_{\text{sup}} \longrightarrow \bar{K}_{\text{sup}}$ such that the nonoptimality measure of the support $\mu(K_{\text{sup}})$ [14] is decreased. The dual method for solving problem (1.6) is obtained on the basis of the support change procedure in the direct method presented in Section 4. In the procedure mentioned the elements $j_0$ and $t_0$ are constructed in a new way:

$$\rho_{j_0} = \max_{j \in J_{\text{sup}}}\rho_j, \quad \rho(t_0) = \max_{j \in T_n}\rho(t);$$

$$\rho_0 = \max\{\rho_{j_0}, \rho(t_0)\}, \quad \rho_j = \rho(\kappa_j, [d_{*j}, d^*_j]),$$
$$j \in J; \quad \rho(t) = \rho(\zeta(t), [\xi_*, \xi^*]), \quad t \in T_h.$$

The maximum of the function $\rho(t)$, $t \in T_h$, can be attained only at the points of extremum of the output pseudosignal $\zeta(t)$, $t \in T_h$. Therefore, to find the instant $t_0$, it suffices to know the sets $T_0^-$ and $T_0^+$ of points of the minimum and maximum of the output pseudosignal $\zeta(t)$, $t \in T_h$, respectively, as well as the values of the fundamental matrix $F(t)$ at these points. Then, the extremal values of the output pseudosignal are $\zeta(t) = y(t) - c'(t)F(t)F^{-1}(t_*)\kappa$, $t \in T_0 = T_0^- \cup T_0^+$.

Thus, at the beginning of each iteration, the following information is stored in the computer memory: (1) the support $K_{\sup}$; (2) the matrix $D_{(\sup)}$; (3) the support values of the function of potentials $\nu_{\sup}$; (4) the nonsupport components of the vector of estimates $\Delta_n$; (5) the pseudostate $\kappa$; (6) the sets $T_0^+$ and $T_0^-$; and (7) the values of the matrix function $F(t)$ at the instants $t \in T_{\sup} \cup T_0 \cup \{t_*, t^*\}$.

Information (1)–(5) for the next iteration is obtained in the same way as in the direct method (the support change procedure). Let us describe the rules of transformation of the sets $T_0^-$ and $T_0^+$.

The new pseudostate $\bar\kappa$ generates a new output pseudosignal $\bar\zeta(t)$, $t \in T_h$, which can be constructed by integrating the direct system (1.1) with the initial condition $x(t_*) = \bar\kappa$ along the whole interval $T$. However, the purpose of this paper is to construct an algorithm that involves minimal integrations of the direct and the dual systems. This purpose can be achieved by employing at each iteration the values of the output signal not at all the instants $t \in T_h$, but only at the instants $t \in T_0 \cup \{t_*, t^*\}$ when the function $\rho(t)$, $t \in T_h$, attains its maximal values.

The values of the output pseudosignal corresponding to the old and new supports are calculated by the formulas

$$\zeta(t) = y(t) - c'(t)F(t, t_*)\kappa;$$
$$\bar\zeta(t) = y(t) - c'(t)F(t, t_*)\bar\kappa.$$

Therefore, the output signal is changed in the direction $\Delta\zeta(t) = -c'(t)F(t, t_*)(\bar\kappa - \kappa)$.

Along with elements (1)–(8) stored in the memory, we make use of the additional information

$$s(t), \vartheta(t), t \in T_0 \cup \{t_*, t^*\}. \tag{5.1}$$

Calculate numbers (5.1) by the rules

$$s(t) = -1, \text{ if } t \in T_0^- \text{ and } \dot d'(t)(\bar\kappa - \kappa) > 0$$
$$\text{or } t \in T_0^+ \text{ and } \dot d'(t)(\bar\kappa - \kappa) < 0;$$
$$s(t) = 1, \text{ if } t \in T_0^+ \text{ and } \dot d'(t)(\bar\kappa - \kappa) > 0$$
$$\text{or } t \in T_0^- \text{ and } \dot d'(t)(\bar\kappa - \kappa) < 0; \tag{5.2}$$
$$s(t_*) = 1, \quad s(t^*) = -1;$$
$$\vartheta(t) = -\frac{\zeta(t + s(t)h) - \zeta(t)}{\Delta\zeta(t + s(t)h) - \Delta\zeta(t)}$$
$$= -\frac{(c'(t + s(t)h)F(t + s(t)h) - c'(t)F(t))F^{-1}(t_*)\kappa}{(c'(t + s(t)h)F(t + s(t)h) - c'(t)F(t))F^{-1}(t_*)(\bar\kappa - \kappa)}.$$

The numbers $s(t)$, $t \in T_0$, indicate the possible direction of motion of the points of maximum and minimum; $\vartheta(t) \ge 0$ is the step for which the extremum point of the output pseudosignal moves from the point $t$ into $t + s(t)h$ along $\Delta\zeta(t)$, $t \in T_h$. Steps $\vartheta(t) < 0$ are not taken into account.

Information (6) about the extremum points of the output pseudosignal and about the values of the matrix function $F(t)$, $t \in T$, at these points will be transformed by stages. Let the $(i-1)$th stage along $\Delta\zeta(t)$, $t \in T_h$ be accomplished and the information before the $i$th stage have the form $T_0^{-i}$, $T_0^{+i}$, $\vartheta^i(t)$, $t \in T_0^i = T_0^{-i} \cup T_0^{+i}$.

Suppose that all the numbers $\vartheta^i(t)$, $t \in T_0^i$, are different with the exception of, perhaps, the numbers $\vartheta^i(t)$, $\vartheta^i(t+h)$, $t, t+h \in T_0^i \cup \{t_*, t^*\}$, where $s(t-h) = 1$, $s(t+h) = -1$.

Calculate the step

$$\vartheta^i = \vartheta^i(t^i) = \min_{\vartheta^i(t) \ge 0} \vartheta^i(t), \quad t \in T_0^i \cup \{t_*, t^*\}. \tag{5.3}$$

In the case $\vartheta^i \le 1$, transform the information stored in the memory in the following way.

(1) If $t^i \in T_0^i$ is a unique number on which the minimum in (5.3) is attained, then $T_0^{i+1} = (T_0^i \backslash t^i) \cup \{t^i + s(t^i)h\}$; store $F(t^i + s(t^i)h)$ instead of $F(t^i)$; calculate the step $\vartheta(t + s(t)h)$ by formula (5.2).

(2) For $t^i = t_*$ or $t^i = t^*$, set $T_0^{i+1} = T_0^i \cup \{t^i + s(t^i)h\}$. Calculate and store $F(t^i + s(t^i)h)$; set $s(t^i + s(t^i)h) = s(t^i)$. Calculate the step $\vartheta(t + s(t)h)$ according to (5.2). The instant $t^i + s(t^i)h$ is the minimum point of the output pseudosignal if $\zeta(t^i) < \zeta(t^i + s(t^i)h)$, or the maximum point when $\zeta(t^i) > \zeta(t^i + s(t^i)h)$.

(3) If $\vartheta^i = \vartheta^i(t) = \vartheta^i(t+h)$, $s(t) = 1$, $s(t+h) = -1$, $t$, $t + h \in T_0$, then $T_0^{i+1} = (T_0^i \backslash \{t, t+h\}$; delete the values $F(t)$, $F(t+h)$ of the fundamental matrix from the computer memory.

(4) In the case $\vartheta^i = \vartheta^i(t^i) = \vartheta^i(t^i + h)$, $s(t^i) = 1$, $s(t^i + h) = -1$, $t^i = t_*$, delete the point $t^i = h$ from the set $T_0^i$: $T_0^{i+1} = T_0^i \backslash t^i + h$; delete the value $F(t^i + h)$ of the fundamental matrix from the computer memory.

(5) For $\vartheta^i = \vartheta_0^i(t^i) = \vartheta_0^i(t^i + h)$, $s(t^i) = 1$, $s(t^i + h) = -1$, $t^i + h = t^*$, set $T_0^{i+1} = T_0^i \backslash t^i$. Delete the value $F(t^i)$ from the memory.

**Theorem 2.** The method is finite if only regular supports are used at its iterations.

There exists a modification of the dual method [14] that is finite for any problem (1.6).

**On the efficiency of the method.** As an efficiency characteristic of the method for solving the problem of optimal observation (1.5), take the number of complete (along the whole interval $T$) integrations of the direct or dual systems necessary for constructing the extremal ($\varepsilon$-extremal) state [15]. If, in the process of solution, several direct or dual systems are independently integrated, then parallel processors may be used for the calculation and such an integration may be considered as a single one. Take one integration as a unit of labor consumption for the method. This is the labor consumption of the procedures for constructing the support and the output pseudosignal. Therefore, the labor consumption of the preparation to the first iteration, as well as of the identification of the optimal support, is equal to two. The labor consumption of the integration along an interval of length $h$ is $1/N$.

Let $M(k)$ be the number of the extremum points at the beginning of the $k$th iteration. Then, the preparation of the additional information (5.1) has the labor consumption $M(k) + 2/N$. Denote by $\overline{M}(k)$ the number of extremum points involved in the $k$th iteration, by $L_m(k)$ the distance traveled by the $m$th point of extremum at the $k$th iteration, and by $k_*$ the number of iterations until the construction of the $\varepsilon$-extremal state. Then, the labor consumption of all iterations of the method is

$$E = \frac{1}{N}\sum_{k=1}^{k_*}\left(M(k) + 2 + \sum_{m=1}^{\overline{M}(k)} 2L_m(k)\right).$$

The employment of parallel computations allows one to raise the efficiency of the method, because in this case the labor consumption of the preparation of the additional information (5.1) is $1/N$ and

$$1/N \text{ and } E = \frac{1}{N}\sum_{k=1}^{k_*}\left(1 + \sum_{m=1}^{\overline{M}(k)} 2L_m(k)\right).$$

The number $E$ can hardly be analytically expressed in terms of the problem parameters. The efficiency of the method is illustrated by the results of the numerical solution of the example considered in Section 7. The experimental estimation of the efficiency of the method as well as the comparison with other methods are beyond the scope of this paper.

## 6. CONSTRUCTION OF A POSITIONAL SOLUTION

Let us start with the definition of a positional solution to the problem of optimal observation (1.5). Embed problem (1.5) into the family of problems

$$p'x \longrightarrow \max,$$
$$\xi_* \le y(t) + d'(t)x \le \xi^*, \quad t \in T(\tau) = [0, \tau],$$
$$d_* \le x \le d^*,$$

which depends on the instant $\tau \in T_h$ and on the function $y_\tau(\cdot) = (y(t), t \in T(\tau))$ from the family $Y(\tau) = \{y(\cdot): y(t) = c'(t)F(t, t_*)x + \xi(t),\ x \in X_0,\ \xi_* \le \xi(t)\ \xi^*,\ t \in T(\tau)\}$ of possible signals of the measuring device. A pair $(\tau, y_\tau(\cdot))$ is called a position of the observation process. It contains all the information about the behavior of system (1.1) that is available at the instant $\tau$.

Call the functionals

$$x^*(\tau, y_\tau(\cdot)),\ \hat{\alpha}(\tau, y_\tau(\cdot)),\ y_\tau(\cdot) \in Y(\tau),\ \tau \in T_h, \quad (6.1)$$

a positional solution to the problem of optimal observation (1.5) if

$$\hat{\alpha}(\tau, y_\tau(\cdot)) = p'x^*(\tau, y_\tau(\cdot)) = \max_{x \in \hat{X}(\tau, y_\tau(\cdot))} p'x,$$

where $\hat{X}(\tau, y_\tau(\cdot))$ is the posterior distribution of the initial state corresponding to the position $(\tau, y_\tau(\cdot))$.

In nontrivial cases, functionals (6.1) cannot be constructed in a closed (explicit) form. Therefore, following [16], we describe a method for the realization of a positional solution in each particular observation process. It is based on analyzing the employment of the positional solution in the observation process.

Suppose that functionals (6.1) are constructed and system (1.1) is observed on the interval $T(\tau)$. Denote by $x_0^*$ the realized initial state of system (1.1) in the considered process, which is unknown to the observer; by $y_\tau^*(\cdot)$ the signal of the measuring device; and by $x^*(\tau) = x^*(\tau, y_\tau^*(\cdot))$ and $\hat{\alpha}^*(\tau, y_\tau^*(\cdot))$ the extremal state and the estimate for the current position $(\tau, y_\tau^*(\cdot))$. Functionals (6.1) are thus seen to be used in a particular observation process incompletely, only their values along the realized output signals of the measuring device are needed. Call the functions $x^*(\tau)$, $\hat{\alpha}^*(\tau)$, $\tau \in T_h$, the realizations of the positional solution to the problem of optimal observation. If, for any $\tau \in T_h$, the functions $x^*(\tau)$ and $\hat{\alpha}^*(\tau)$ are constructed within a time of $h$ or less, then we say that the positional solution is realized in the real-time mode.

We refer to a device that in each particular observation process can calculate $x^*(\tau)$, $\hat{\alpha}^*(\tau)$, $\tau \in T_h$, in the real-time mode as the optimal estimator.

Let us describe the operation algorithm of an optimal estimator. Before starting the observation process, the extremal possible state $x^*$ and the estimate $\hat{\alpha}^*$ are calculated by the prior distribution $X_0$; i.e., the linear programming problem

$$p'x \longrightarrow \max, \quad x \in X_0 \tag{6.2}$$

is solved. Let $K^*_{\text{sup}}$ be the optimal support of problem (6.2).

Assume that the optimal estimator has been constructed and operated at the instants $t_*$, $t_* + h$, …, $\tau$, having calculated the extremal possible states $x^*(t_*)$, $x^*(t_* + h)$, …, $x^*(\tau)$; the estimates $\hat{\alpha}^*(t_*)$, $\hat{\alpha}^*(t_* + h)$, …, $\hat{\alpha}^*(\tau)$ on the basis of the realized signals $y^*(t_*)$, $y^*(t_* + h)$, …, $y^*(\tau)$. At the instant $\tau + h$, the estimator gets the signal $y^*(\tau + h)$ and has to quickly calculate the values $x^*(\tau + h)$ and $\hat{\alpha}^*(\tau + h)$. By assumption, when calculating the values $x^*(\tau)$ and $\hat{\alpha}^*(\tau)$ at the instant $\tau$, the estimator has already solved the problem

$$\begin{gathered} p'x \longrightarrow \max, \quad \xi_* \le y^*(t) + d'(t)x \le \xi^*, \\ t \in T(\tau) = \{t_*, t_* + h, \dots, \tau\}; \quad d_* \le x \le d^*, \end{gathered} \tag{6.3}$$

and it possesses the following information: (1) the optimal support $K^*_{\text{sup}}(\tau)$; (2) the matrix $D_{(\text{sup})}(\tau)$; (3) the support values of the function of potentials $\nu^*_{\text{sup}}(\tau)$; (4) the vector of estimates $\Delta^*(\tau)$; (5) the pseudostate $\kappa^*(\tau)$; (6) the sets $T_0^+(\tau)$ and $T_0^-(\tau)$; (7) the values of the matrix function $F(t)$ at the instants $t \in T_0(\tau) \cup \{t_*, \tau - h, \tau\}$.

The problem to be solved by the optimal estimator at the instant $\tau + h$ differs from problem (6.3) by the presence of an additional constraint at the instant $\tau + h$ and has the form

$$\begin{gathered} p'x \longrightarrow \max, \quad \xi_* \le y^*(t) + d'(t)x \le \xi^*, \quad t \in T(\tau); \\ \xi_* \le y^*(\tau + h) + d'(\tau + h)x \le \xi^*; \quad d_* \le x \le d^*. \end{gathered} \tag{6.4}$$

To solve problem (6.4), let us take the optimal support $K_{\text{sup}}(\tau)$ of problem (6.3) as the initial support.[4] Then, $\nu_{\text{sup}}(\tau + h) = \nu^*_{\text{sup}}(\tau)$; $\Delta(\tau + h) = \Delta^*(\tau)$; $\kappa(\tau + h) = \kappa^*(\tau)$. To calculate the value of the output pseudosignal at the instant $\tau + h$, let us integrate the equation $\dot{F} = A(t)F$ along the interval $[\tau, \tau + h]$ with the known initial condition $F(\tau)$. Then, $\zeta(\tau + h) = y(\tau + h) - c'(\tau + h)F(\tau + h)F^{-1}(t_*)\kappa(\tau + h)$.

If the inequalities $\xi_* \le \zeta(\tau + h) \le \xi^*$ hold, then $K^*_{\text{sup}}(\tau)$ is the optimal support of problem (6.4). Information (1)–(7) written for the instant $\tau$ has to be rewritten for the instant $\tau + h$. Otherwise, construct the optimal support $K^*_{\text{sup}}(\tau + h)$ by the dual method presented in Section 5. In both cases, having the values $F(\tau - h)$, $F(\tau)$, and $F(\tau + h)$, we calculate the values of the output pseudosignal $\zeta^*(\tau - h)$, $\zeta^*(\tau)$, and $\zeta^*(\tau + h)$ companion for the optimal support $K^*_{\text{sup}}(\tau + h)$. By these values, we find out whether the instant $\tau$ is an extremum of the output pseudosignal and include it in the set $T_0(\tau + h)$.

The extremal possible state and the estimate at the instant $\tau + h$ are equal to $x^*(\tau + h) = \kappa^*(\tau + h)$ and $\hat{\alpha}^*(\tau + h) = p'x^*(\tau + h)$.

The actual practice of employing the dual method (and the results of the experiment carried out) demonstrates that the estimator that realizes the above-described algorithm by means of modern microprocessors can solve problems of optimal observation of dynamical systems of sufficiently high order.

## 7. EXAMPLE

Consider an example of observing an oscillatory two-mass system (Fig. 1) on the interval $T = [0, 5]$. The mathematical model of the system has the form

$$\begin{gathered} \dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -x_1 + x_2, \\ \dot{x}_4 = 0.1x_1 - 1.02x_2. \end{gathered} \tag{7.1}$$

Let the prior distribution of the initial state $X_0$ be defined by the relations[5]

$$x_1 = 0, \quad x_2 = 0, \quad |x_3| \le 1, \quad |x_4| \le 1,$$

and the measuring device measure the position of the first mass $x_1(t)$ with a bounded error $|\xi(t)| \le 0.25$, $t \in T_h$; i.e., the measuring device provides the signal $y(t) =$

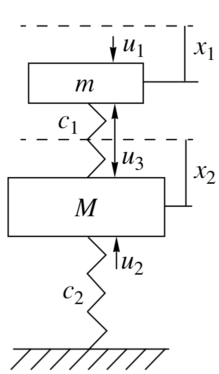


**Fig. 1.** Two-mass oscillatory system.

[4] If $\tau = t_*$, then take $K^*_{\text{sup}}$ as the initial support.

[5] At the starting instant, the rest masses may be exposed to blows of a bounded force.

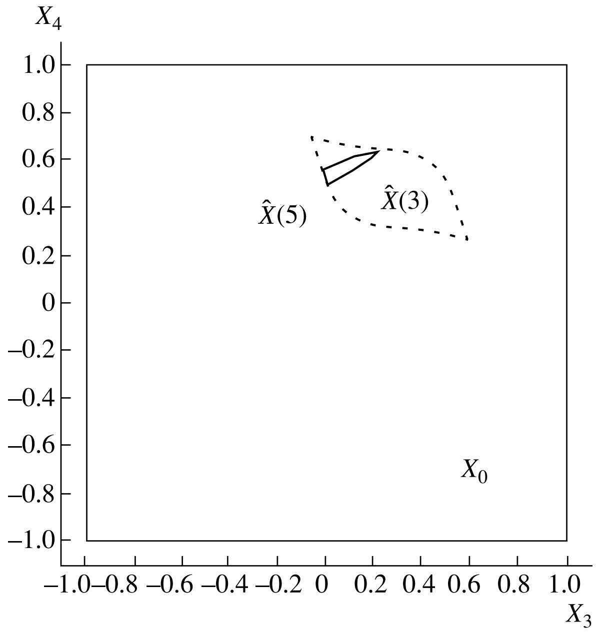


**Fig. 2.** Prior and posterior distributions.

$x_1(t) + \xi(t)$ at the discrete instants $t \in T_h = \{0, h, 2h, \dots, 5 - h, 5\}$, $h = 0.005$.

Assume that the true (but unknown) initial state of the system is.[6]

$$x_1(0) = 0, \quad x_2(0) = 0, \quad x_3(0) = 0, \quad x_4(0) = 0.5,$$

and the realized (but unknown) error function of the measuring device has the form

$$\xi(t) = \begin{cases} 0.25\sin t, & t \in [0, 4[; \\ 0.25\sin 2t, & t \in [4, 5]. \end{cases}$$

In Fig. 2, the prior set $X_0$ and the boundary points of the posterior set $\hat{X}(3)$ constructed by the observation results on the interval [0, 3] and of the set $\hat{X}(5)$ constructed by the observation results on the interval [0, 5] are presented. These points are obtained as a result of calculating the extremal states in the directions $p = (0, 0, \cos\phi, \sin\phi)$, $\phi = k\pi/18$, $k = \overline{0.36}$, by the dual method.

Let us present the results of the posterior solution by the dual method for two opposite directions. For the vector $p = (0, 0, 0, 1)$, the optimal support $K_{\sup} = \{T_{\sup}, J_{\sup}\}$ has the components $T_{\sup} = \{3.48; 4.265\}$, $J_{\sup} = \{3, 4\}$, the estimate is equal to $\hat{\alpha}^* = 0.63328$, the extremal initial state has the form $x^* = (0; 0; 0.20774; 0.63328)$, and the corresponding output signal is shown in Fig. 3. The efficiency of the iterations for an empty initial support is $E = 2.241$.

[6] Only the second mass was exposed to a blow of nonmaximal force.

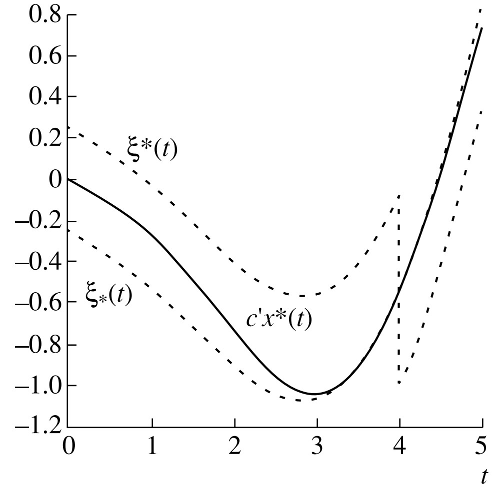


**Fig. 3.** Solution to the problem of optimal observation for $p = (0, 0, 0, 1)$.

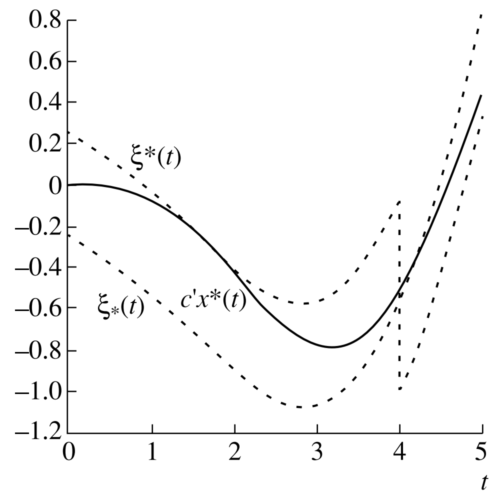


**Fig. 4.** Solution to the problem of optimal observation for $p = (0, 0, 0, -1)$.

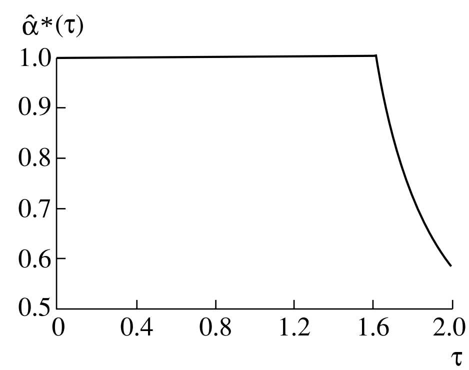


**Fig. 5.** Behavior of the estimate for the posterior distribution for the positional solution.

For $p = (0, 0, 0, -1)$, we have $T_{\sup} = \{1.575; 4.0\}$, $J_{\sup} = \{3, 4\}$, $\hat{\alpha}^* = -0.49794$, $x^* = (0; 0; 0.00102; 0.49794)$ (Fig. 4), and $E = 2.294$.

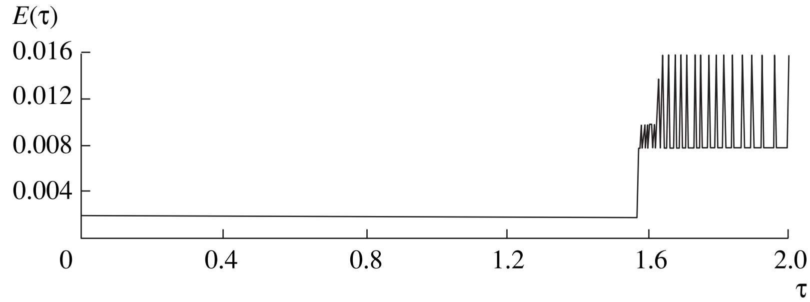


**Fig. 6.** Efficiency of the algorithm for constructing a positional solution.

Now, pass to the construction of the positional solution. System (7.1) is observed on the interval $T = [0, 2]$; $p = (0, 0, 0, 1)$, $\xi(t) = 0.25\sin 2t$, $t \in [0, 2[$. In Fig. 5, the function $\hat{\alpha}^*(\tau)$, $\tau \in T_h$, is shown. The information about the labor consumption of the procedure for correction of the supports by the dual method for $h = 0.004$ is presented in Fig. 6.

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