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# Design of Optimal Feedbacks in the Class of Inertial Controls[1]

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**Abstract**—Two optimal control problems in the class of inertial controls (continuous controls of bounded values and rates) are investigated: the first is the optimal excitation of dynamic systems and the related design of attainability sets and the second is a linear optimal control problem with terminal constraints. They are solved by a dynamic adaptive linear programming method. First, open-loop solutions are constructed and then an algorithm for the optimal controller for the linear optimal control problem is designed, which in real time generates current optimal feedback. Examples are given to illustrate the results.

## 1. INTRODUCTION

The elegance of the formulation of the Pontryagin maximum principle [1] in optimal control problems without phase constraints is mostly attributed to the wide classes of admissible controls used in the formulation of problems. These classes—piecewise-continuous or measurable functions—are mathematical models of inertialess controls, which cannot be realized in pure form. Real control actions are bounded not only in values, but also in rate of variation of values. The use of such constraints results in optimal control problems with phase constraints. The present qualitative theory (in particular, the theory of necessary conditions of optimality) for optimal control problems has been elaborated to the same depth [2] as the similar theory for optimal control problems without phase constraints. But the maximum principle for optimal control problems with phase constraints is not as elegant as the classical Pontryagin maximum principle and, in general, does not reduce the initial extremal problem to a finite-dimensional problem. There are only a few examples in which the optimal control structure yields with great difficulty to analysis by the maximum principle. Therefore, at present there is no hope for effectively constructing even open-loop controls for optimal control problems in the class of inertial controls with the help of well-known necessary conditions of optimality. The design of optimal feedback controls for problems with phase constraints has almost not been touched upon in the literature.

In this paper, we design constructive methods of solving optimal control problems in the class of inertial controls based on dynamic realization of the adaptive linear programming method [3]. Earlier [4], the adaptive method was successfully applied to optimal control problems without phase constraints in the class of discrete controls. Now as a continuation of the approach of [4] the adaptive method is extended to a new class of optimal control problems. Furthermore, the main attention is oriented on the maximal use of the specifics of phase constraints. This aids not only in proving the optimality and suboptimality criteria and designing fast computation algorithms for open-loop controls, but also in designing optimal feedback controls.

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The paper begins (Section 2) with the solution to the problem of optimal excitation of dynamic systems—the simplest optimal control problem in the class of inertial controls. This problem is useful in distinctly delineating the operations involved in the main construct of the paper—the support. We shall show that if the specifics of constraints of our problem are adequately taken into account, then the adaptive linear programming method can be realized dynamically (the maximum and $\varepsilon$-maximum principles can be proved in terms of the adjoint system, effective computation of optimal open-loop controls is demonstrated). The method thus constructed is close in complexity to the method of computing optimal open-loop controls in a linear optimal control problem without phase constraints [4]. In Section 3, we construct attainability sets for dynamic systems—an important problem in many respects. We describe a support correction method, which aids in quickly constructing arbitrarily exact approximations for attainability sets. This method is applied in Sections 4 and 5 to compute open-loop controls and design optimal feedback controls in linear optimal control problems in the class of inertial controls. Formulation and an open-loop control for a linear optimal control problem are given in Section 4. After eliminating terminal constraints, the problem is solved by the dual method with the help of the Lagrange vector. For this purpose, a special initial approximation for the Lagrange vector, an iterative method for improving the approximation, and an optimization method to construct an open-loop control to a linear optimal control problem are designed. Section 5 is devoted to the design of optimal feedback controls through the approach of [4] as applied to a simpler problem. Since the optimal feedback for our problem cannot be constructed in closed (explicit) form, it is implemented with an optimal controller. An algorithm for the operation of the optimal controller is described. It generates an optimal feedback in every control cycle in real time. An example on the optimal control of an oscillatory system with one degree of freedom is given to illustrate the results of the paper.

## 2. OPTIMAL EXCITATION OF DYNAMIC SYSTEMS

Let $T = [t_*, t^*]$ be the control interval. A scalar function $u(t)$, $t \in T$, is said to be the inertial control action if it is the control $\dot{u} = v$, $u(t_*) = u_0$, with bounded piecewise-continuous function $v(t)$, $t \in T$.

Since discrete computing devices are required to solve most of applied optimal control problems, we assume that the control signal $v(t)$, $t \in T$, is a discrete function with quantization period $h = (t^* - t_*)/N$, where $N$ is an integer:

$$v(t) = v(t_* + kh), \quad t \in [t_* + kh, t_* + (k+1)h[\,, \quad k = \overline{0, N-1}.$$

For the sake of simplicity, each of the functions in a pair of functions of control action $u(t)$, $t \in T$, and control signal $v(t)$, $t \in T$, is called the control.

Let $S = T$, $S_h = \{t_* + h, t_* + 2h, \dots, t^*\}$, and $T_h = \{t_*, t_* + h, \dots, t^* - h\}$. An inertial control is said to be admissible if it satisfies the constraints $|u(s)| \le L$, $s \in S$, and $|v(t)| \le M$, $t \in T$. The set of admissible inertial controls $u(\cdot) = (u(t), t \in T)$ is denoted by $U^1$.

In the class of inertial controls, let us consider the optimal excitation problem

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \quad u(\cdot) \in U^1, \tag{1}$$

where $x = x(t)$ is the $n$-dimensional state vector of the dynamic system at instant $t$, $u = u(t)$ is the scalar control action, and $A(t)$ and $b(t)$, $t \in T$, are piecewise-continuous $n \times n$ matrix and $n$-vector functions, respectively.

Introducing a supplementary phase variable $x_{n+1} = u$, we can regard problem (1) as a problem with phase constraint

$$\begin{gathered}\overline{c}'\overline{x}(t^*) \to \max, \quad \dot{\overline{x}} = \overline{A}(t)\overline{x} + \overline{b}(t)v, \quad \overline{x}(t_*) = (x_0, u_0), \\ |x_{n+1}(s)| \le L, \quad s \in S, \quad |v(t)| \le M, \quad t \in T,\end{gathered} \tag{2}$$

where

$$\overline{x} = (x, x_{n+1}), \quad \overline{A}(t) = \begin{pmatrix} A(t) & b(t) \\ 0 & 0 \end{pmatrix}, \quad \overline{b}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \overline{c} = (c, 0).$$

Such a form of the problem will prove useful in the sequel.

An admissible control action $u^0(\cdot)$ and its corresponding control signal $v^0(\cdot)$ are said to be optimal if $c'x^0(t^*) = \max c'x(t^*)$, $u(\cdot) \in U^1$, on the (optimal) trajectory $x^0(t)$, $t \in T$, they generate.

For a given $\varepsilon \ge 0$, the suboptimal ($\varepsilon$-optimal) controls $u^\varepsilon(t)$, $v^\varepsilon(t)$, $t \in T$, and the trajectory $x^\varepsilon(t)$, $t \in T$, are defined by the inequality $c'x^0(t^*) - c'x^\varepsilon(t^*) \le \varepsilon$.

Problem (1) is trivial if one of the numbers $L$ and $M$ is infinite.

Using the Cauchy formula (see A.1), we can easily show that problem (1) in the class of admissible controls is equivalent to the linear programming problem

$$\sum_{t \in T_h} c(t)v(t) \to \max, \quad -L \le h\sum_{t=0}^{s-h} v(t) \le L, \quad s \in S_h, \quad -M \le v(t) \le M, \quad t \in T_h, \tag{3}$$

where $c(t) = \int_t^{t+h} \psi_{n+1}(\tau)d\tau = \psi^h_{n+1}(t)h$, $t \in T_h$, and $\psi_{n+1}(t)$, $t \in T$, is the component of the solution $(\psi(t), \psi_{n+1}(t))$, $t \in T$, of the adjoint system

$$\dot{\psi} = -A'(t)\psi, \quad \dot{\psi}_{n+1} = -\psi'b(t) \tag{4}$$

under the initial conditions

$$\psi(t^*) = c, \quad \psi_{n+1}(t^*) = 0, \tag{5}$$

where $\psi^h_{n+1}(t) = \int_t^{t+h} \psi_{n+1}(\tau)d\tau/h$, $t \in T_h$, is the mean Steklov function [5] and $\psi^h_{n+1}(t) \to \psi_{n+1}(t)$ as $h \to 0$.

As stated in the Introduction, our results are based on dynamic realization of the adaptive method [3] for problem (3). Necessary details of this method are given in Appendix 2.

Let us begin with the basic concept—the support. From the set $S_h$ let us take an arbitrary ensemble of disjoint segments $S_l = \left\{s_l, s_l + h, \ldots, s^l\right\}, l = \overline{1, l^*}$. Let us associate every $S_l$ with a segment $T_l = \{t_l, t_l + h, \ldots, t^*\}$, $t_l = s_l$, $t^l = s^l - h$, and an instant $\tau_l$, $s^{l-1} \le \tau_l < s_l$. Let $S_{\sup} = \left\{S_l, l = \overline{1, l^*}\right\}$, $T^N_{\sup} = \left\{T_l, l = \overline{1, l^*}\right\}$, $T^R_{\sup} = \left\{\tau_l, l = \overline{1, l^*}\right\}$, and $T_{\sup} = T^N_{\sup} \bigcup T^R_{\sup}$.

**Lemma.** *The set $K_{\sup} = \{S_{\sup}, T_{\sup}\}$ is the support for problem* (1). *Every support of problem* (1) *is of the structure described above.*

A pair $K_{\sup} = \{S_{\sup}, T_{\sup}\}$ of empty sets $S_{\sup} = \varnothing$ and $T_{\sup} = \varnothing$ is an empty support $K_{\sup}$ by definition. If $S_{\sup} = S_h$, $T^R_{\sup} = \left\{s^l, l = \overline{1, l^*}\right\}$, and $T^N_{\sup} = T_h \setminus T^R_{\sup}$, then $K_{\sup} = \{S_{\sup}, T_{\sup}\}$ is the maximal support.

Every support $K_{\text{sup}}$ has the following:

(1) A potential function (dynamic analog of the potential vector) $\nu(s)$, $s \in S_h$:

$$\begin{gathered}
\nu\left(s^{l^*}\right) = c\left(s^{l^*} - h\right)/h, \quad \nu(s) = (c(s-h) - c(s))/h, \quad s = \overline{s_l + h, s^l - h}, \quad l = \overline{1, l^*}, \\
\nu(s_l) = (c(\tau_l) - c(t_l))/h, \quad l = \overline{1, l^*}, \\
\nu\left(s^l\right) = \left(c\left(s^l - h\right) - c(\tau_{l+1})\right)/h, \quad l = \overline{1, l^* - 1}, \\
\nu(s) = 0, \quad s \in S_{\mathrm{N}}.
\end{gathered} \tag{6}$$

It has the following properties:

$$\begin{gathered}
\nu\left(s^{l^*}\right) \to \psi_{n+1}\left(s^{l^*} - h\right), \quad h \to 0, \\
\nu\left(s^l\right) \to \psi_{n+1}\left(s^l - h\right) - \psi_{n+1}(\tau_{l+1}), \quad h \to 0, \quad l = \overline{1, l^* - 1}, \\
\nu(s_l) \to \psi_{n+1}(\tau_l) - \psi_{n+1}(s_l), \quad h \to 0, \quad l = \overline{1, l^*}, \\
\nu(s) \to 0, \quad h \to 0, \quad s = \overline{s_l + h, s^l - h}, \quad l = \overline{1, l^*}.
\end{gathered}$$

The potential function $\nu(s)$, $s \in S_h$, for an empty set is not computed (we assume that $\nu(s) \equiv 0$, $s \in S_h$).

(2) A cocontrol (analog of the estimate vector):

$$\begin{aligned}
&\Delta_h(t) = c(t)/h, \quad t > t^{l^*}, \\
&\Delta_h(t) = (c(t) - c(\tau_l))/h, \quad t^{l-1} < t < t_l, \quad l = \overline{1, l^*}, \\
&\Delta_h(t) = 0, \quad t \in T_{\text{sup}}.
\end{aligned} \tag{7}$$

The cocontrol has the following properties:

$$\begin{gathered}
\Delta_h(t) \to \psi_{n+1}(t), \quad h \to 0, \quad t > t^{l^*}, \\
\Delta_h(t) \to \psi_{n+1}(t) - \psi_{n+1}(\tau_l), \quad t^{l-1} < t < t_l, \quad l = \overline{1, l^*}.
\end{gathered}$$

For an empty support, $\Delta_h(t) = c(t)/h$, $t \in T_h = T_{\mathrm{N}}$.

A support is said to be regular if $\nu(s) \neq 0$, $s \in S_{\text{sup}}$ and $\Delta_h(t) \neq 0$, $t \in T_{\mathrm{N}}$.

(3) A pseudosignal $\omega(t)$, $t \in T$, and a pseudoaction $\zeta(s), s \in S$. First, we construct $\omega(t)$, $t \in T_{\mathrm{N}}$, and $\zeta(s), s \in S_{\text{sup}}$:

$$\begin{gathered}
\omega(t) = M \operatorname{sgn} \Delta_h(t) \quad \text{for} \quad \Delta_h(t) \neq 0, \\
\omega(t) \in [-M, M], \quad \text{for} \quad \Delta_h(t) = 0, \quad t \in T_{\mathrm{N}},
\end{gathered} \tag{8}$$

$$\begin{gathered}
\zeta(s) = L \operatorname{sgn} \nu(s), \quad \text{for} \quad \nu(s) \neq 0, \\
\zeta(s) \in [-L, L], \quad \text{for} \quad \nu(s) = 0, \quad s \in S_{\text{sup}}.
\end{gathered} \tag{9}$$

Then we shall find $\omega(t), t \in T_{\text{sup}}$, and $\zeta(s), s \in S_{\mathrm{N}}$ from

$$\zeta(s) = u_o + h \sum_{t=0}^{s-h} \omega(t), \quad s \in S_h.$$

The functions $\omega(t)$, $t \in T_h$, and $\nu(s)$, $s \in S_h$, are defined on $T$ and $S$, respectively.

The accompanying pseudosignal and pseudoaction are unique if the support $K_{\text{sup}}$ is regular.

Let us describe a dynamic method of constructing potential functions $\nu(s)$ (6), $s \in S_h$, and cocontrols $\Delta_h(t)$ (7), $t \in T_h$, from the support $K_{\text{sup}} = \{S_{\text{sup}}, T_{\text{sup}}\}$. Let $s \in S_{\text{sup}}$ and $\mu_*(s) \in T_{\text{sup}}$ be the nearest support instant on the left of $s$ and let $\mu^*(s) \in T_{\text{sup}}$ be the nearest or coincident support instant on the right of $s$. Then, by virtue of the formula $c(t)/h = \psi^h_{n+1}(t)$, $t \in T_h$, from (6) and (7) we obtain

$$\begin{gathered} \nu\left(s^{l^*}\right) = \psi^h_{n+1}\left(\mu_*\left(s^{l^*}\right)\right), \\ \nu(s) = \psi^h_{n+1}(\mu_*(s)) - \psi^h_{n+1}(\mu^*(s)), \quad s \in S_{\text{sup}} \setminus s^{l^*}, \\ \nu(s) = 0, \quad s \in S_{\text{N}}, \end{gathered} \tag{10}$$

$$\Delta_h(t) = \psi^h_{n+1}(t) - \sum_{s=t+h}^{t^*} \nu(s), \quad t \in T_{\text{N}}, \quad \Delta_h(t) = 0, \quad t \in T_{\text{sup}}. \tag{11}$$

Thus the values of functions (10) and (11) accompanying the support $K_{\text{sup}} = \{S_{\text{sup}}, T_{\text{sup}}\}$ can be determined by right and left integrating the adjoint system (4), (5).

Besides computing $\nu(s)$, $s \in S_h$, and $\Delta_h(t)$, $t \in T_h$, let us construct $\omega(t)$ (8), $t \in T_{\text{N}}$, and $\zeta(s)$ (9), $s \in S_{\text{sup}}$. The support values of the pseudosignal $\omega(t)$, $t \in T_{\text{sup}}$, are chosen such that they generate, along with nonsupporting values of $\omega(t)$ (8), $t \in T_{\text{N}}$, a pseudoaction $\zeta(t)$, $t \in T$, which is a continuous line starting from $u_0$ ($\zeta(t_*) = u_0$) and passing through the values of $\zeta(s)$ (9), $s \in S_{\text{sup}}$.

**Theorem 1** (maximum principle). *For admissible control action $u(\cdot)$ and control signal $v(\cdot)$ to be optimal, it is necessary and sufficient that there exist a support $K_{\text{sup}} = \{S_{\text{sup}}, T_{\text{sup}}\}$ such that the conditions of*

*maximum action*

$$\nu(s)u(s) = \max_{-L \le u \le L} \nu(s)u, \quad s \in S_{\text{sup}}, \tag{12}$$

*and maximum signal*

$$\Delta_h(t)v(t) = \max_{-M \le v \le M} \Delta_h(t)v, \quad t \in T_{\text{N}} \tag{13}$$

*are satisfied for certain accompanying potential function $\nu(s)$, $s \in S_h$, and cocontrol $\Delta_h(t)$, $t \in T_h$.*

**Theorem 2** ($\varepsilon$-maximum principle). *For an admissible control $u(\cdot)$ to be $\varepsilon$-optimal for any $\varepsilon \ge 0$, it is necessary and sufficient that there exist a support $K_{\text{sup}} = \{S_{\text{sup}}, T_{\text{sup}}\}$ such that the conditions of*

*$\varepsilon$-maximum action*

$$\nu(s)u(s) = \max_{-L \le u \le L} \nu(s)u - \varepsilon_u(s), \quad s \in S_{\text{sup}}, \tag{14}$$

*$\varepsilon$-maximum signal*

$$\Delta_h(t)v(t) = \max_{-M \le v \le M} \Delta_h(t)v - \varepsilon_v(t), \quad t \in T_{\text{N}}, \tag{15}$$

*$\varepsilon$-accuracy*

$$\sum_{s \in S_{\text{sup}}} \varepsilon_u(s) + \sum_{t \in T_{\text{N}}} \varepsilon_v(t) \le \varepsilon \tag{16}$$

*are satisfied for certain accompanying potential function $\nu(s)$, $s \in S_h$, and cocontrol $\Delta_h(t)$, $t \in T_h$.*

*A support on which (12) and (13) are satisfied is said to be optimal.*

**Theorem 3** (support optimality criterion). *For a support $K_{\text{sup}}$ to be optimal, it is necessary and sufficient that on some accompanying pseudoaction and pseudosignal*

$$|\zeta(s)| \leq L, \quad s \in S_{\text{N}}, \quad |\omega(t)| \leq M, \quad t \in T_{\text{sup}}. \tag{17}$$

The pseudosignal $\omega(t)$, $t \in T$, and the pseudoaction $\zeta(s)$, $s \in S$, accompanying the optimal support are optimal: $v^0(t) = w(t)$, $t \in T$, and $u^0(s) = \zeta(s)$, $s \in S$.

**Corollary 1.** *An empty support is optimal if some accompanying pseudoaction $\zeta(s)$, $s \in S_h$, satisfies the inequality*

$$|\zeta(s)| \leq L, \quad s \in S_{\text{N}}.$$

Obviously, this inequality is satisfied for problem (1) with a sufficiently large $L$ and small $M$. In this case, the optimal signal $v^0(t), t \in T$, coincides with the optimal control in the class of discrete controls for the problem

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)x_{n+1}, \quad \dot{x}_{n+1} = u, \quad x(t_*) = x_0,$$
$$x_{n+1}(t_*) = u_0, \quad |u(t)| \leq M, \quad t \in T.$$

**Corollary 2.** *The maximal support is optimal if $Mh \geq 2L$.*

The optimal action $u^0(t)$, $t \in T$, for the maximal support is close in structure to the optimal control in the class of discrete controls for the problem

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \quad |u(t)| \leq L, \quad t \in T,$$

and differs from the latter only in continuity.

Let us describe a *dual method* of constructing the optimal open-loop control for problem (1). We assume that $Mh < 2L$, otherwise the solution would be the pseudoaction accompanying the maximal support.

The method is simply a sequential transformation of supports for constructing an optimal support. Any support can be used as the initial support. In what follows, the initial support is an empty support.

For the initial support $K_{\text{sup}}$, let us construct its accompanying elements $\nu(s) \equiv 0$, $s \in S_h$; $\Delta_h(t)$, $t \in T_h$; $\omega(t)$, $t \in T$; and $\zeta(s)$, $s \in S$. In constructing these elements, the following information must be stored for use in the sequel: (1) zeros of the cocontrol ($\mu \in T_h$ is a zero of the cocontrol if $\Delta_h(\mu)\Delta_h(\mu - h) < 0, \Delta_h(\mu) \neq 0$), (2) values of the cocontrol at control interval endpoints and at extremum points ($\mu \in T_h$ is the point of maximum if $\Delta_h(\mu - h) < \Delta_h(\mu) \geq \Delta_h(\mu + h)$, $\Delta_h(\mu) < 0$; $\mu \in T_h$ is the point of minimum if $\Delta_h(\mu - h) > \Delta_h(\mu) \leq \Delta_h(\mu + h), \Delta_h(\mu) > 0$), (3) segments from which the pseudoaction escapes from the interval $[-L, L]$, (4) points of extremum of the pseudoaction between the components of the set $S_{\text{sup}}$, and (5) values of $\zeta(t^*)$.

Assume that $K_{\text{sup}}$ is not an optimal support. From among all possible accompanying pseudoactions $\zeta(s)$, $s \in S$, let us choose the pseudoaction for which $|\zeta(t^*)|$ is minimal. The method consists of three steps: (I) first iteration, (II) correction of $\zeta(s)$, $s \in S$, and (III) correction of $\omega(t)$, $t \in T$.

(I) Let $s_*$ be the first right point at which $|\zeta(s_*)| > L$. For the sake of definiteness, assume that $\zeta(s_*) > L$ (the case $\zeta(s_*) < -L$ is investigated along similar lines).

The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_h \setminus s_*. \tag{18}$$

The corresponding variation of the cocontrol is

$$\Delta\delta_h(t) = -1, \quad t \in T_h. \tag{19}$$

In the course of motion of

$$\begin{gathered}\nu(s|\sigma) = \nu(s) + \sigma\Delta\nu(s), \quad s \in S_h,\\ \Delta_h(t|\sigma) = \Delta_h(t) + \sigma\Delta\delta_h(t), \quad t \in T_h, \quad \sigma \geq 0,\end{gathered} \tag{20}$$

from $(\nu(\cdot), \Delta_h(\cdot))$ along the directions of (18) and (19), the initial rate $\alpha^1$ of decrement of the dual aim function is $L - \zeta(s_*) < 0$.

With increasing $\sigma \geq 0$, new zeros

$$\begin{gathered}\sigma_u(s) = \begin{cases} -\nu(s)/\Delta\nu(s) & \text{if} \;\; \nu(s)\Delta\nu(s) < 0\\ \infty & \text{if} \;\; \nu(s)\Delta\nu(s) \geq 0, \quad s \in S_{\text{sup}},\end{cases}\\ \sigma_v(t) = \begin{cases} -\Delta_h(t)/\Delta\delta_h(t) & \text{if} \;\; \Delta_h(t)\Delta\delta_h(t) < 0\\ \infty & \text{if} \;\; \Delta_h(t)\Delta\delta_h(t) \geq 0, \quad t \in T_{\text{N}},\end{cases}\end{gathered} \tag{21}$$

of functions (20) appear. They (their location and number) for small $h$ virtually do not depend on $h$. They may appear only in the neighborhoods of zeros at the points of local minimum of the cocontrol $\Delta_h(t)$, $t \in T_h$. Zeros may vanish at the points of local maximum cocontrol. If the values of the cotrajectory $\psi(t)$, $t \in T$, at these points are stored in the memory, we can quickly compute number (20) by integrating the adjoint system (4) on small segments. This property of the dynamic adaptive method was first used in [4] and greatly improves the effectiveness of dynamic realization compared to the adaptive method used in linear programming problems (3).

To avoid complicated description of degenerate cases, we only confine ourselves to a regular situation in which the cocontrol between adjacent zeros has no local minima on segments of positive values and no local maxima on segments of negative values. In the motion of (20), the dual aim function between adjacent zeros of the cocontrol $\sigma^k < \sigma^{k+1}$ (21) varies linearly with $\sigma$. Its rate $\alpha^k$ experiences a jump $\Delta\alpha^k = 2Mh$, i.e., $\alpha^{k+1} = \alpha^k + 2Mh$, at the point $\sigma^{k+1}$ on this segment.

We shall increase $\sigma$ in (20) until $\alpha^{k_0} < 0, \alpha^{k_0+1} \geq 0$. The superscript $k_0 < \infty$ necessarily exists, because the constraints of problem (1) are consistent. For $\alpha^1 \leq 2Mh$, we obtain $k_0 = 1$ (a short dual step is implemented). Otherwise, a long dual step is implemented. Let $\sigma^* = \sigma^{k_0}$ and let $\tau_*$ be the instant at which the step $\sigma^* : \Delta_h(\tau_* \mid \sigma^*) = 0$ is realized.

Let us construct a new support $\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = s_*, \overline{T}_{\text{sup}} = \tau_*\right\}$. It is accompanied with

$$\begin{gathered}\overline{\nu}(s_*) = \nu(s_*|\sigma^*) = \Delta_h(\tau_*), \quad \overline{\nu}(s) = \nu(s|\sigma^*) = 0, \quad s \in S_h \setminus s_*,\\ \overline{\Delta}_h(t) = \Delta_h(t|\sigma^*) = \Delta_h(t) - \Delta(\tau_*), \quad t \in T_h \setminus \tau_*, \quad \overline{\Delta}_h(\tau_*) = \Delta(\tau_*|\sigma^*) = 0.\end{gathered} \tag{22}$$

Using (22), let us compute $\overline{\zeta}(s)$, $s \in S_h$, and $\overline{\omega}(t)$, $t \in T$, that accompany the support $\overline{K}_{\text{sup}}$. In computations, appropriate corrections for information are introduced.

This concludes the operations involved in eliminating the "outlier" $\zeta(s_*)$. We now go to step (II), using the support $\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = s_*,\, \overline{T}_{\text{sup}} = \tau_*\right\}$.

(II) Let $s_* \in S_h$ be the first right instant at which $|\zeta(s_*)| > L$. For the sake of definiteness, we assume that $\zeta(s_*) > L$ (the case $\zeta(s_*) < -L$ is investigated along similar lines). There are eight possible cases: (1) $s_* = s_l - h$, $\tau_l \neq t_l - h$, $l = \overline{2, l^*}$, (2) $s_* = s_l - h$, $\tau_l = t_l - h$, $l = \overline{2, l^*}$, (3) $s_* = s_1 - h$, $\tau_1 \neq t_1 - h$, (4) $s_* = s_1 - h$, $\tau_1 = t_1 - h$, (5) $s_* = s^l + h$, $l = \overline{1, l^* - 1}$, (6) $s_* = t^*$, (7) $s_* < s_1$, $\tau_1 < s_*$, and (8) $s_* < s_1$, $\tau_1 \geq s_*$.

(1) $\left(s_* = s_l - h, \tau_l \neq t_l - h, l = \overline{2, l^*}\right)$. The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\mathrm{N}} \setminus s_*. \tag{23}$$

The corresponding variation of the cocontrol is

$$\Delta\delta_h(t_l - h) = 1, \quad \Delta\delta_h(t) = 0, \quad t \in T_h \setminus \{t_l - h\}, \tag{24}$$

and the variation of the support component of the potential function is

$$\Delta\nu(s_l) = -1, \quad \Delta\nu(s) = 0, \quad s \in S_{\mathrm{sup}} \setminus s_l. \tag{25}$$

Properties (24) and (25) characterize the second feature of the dynamic adaptive method. They arise due to the specific phase constraints of problem (1). Consequently, the potential function is not computed at the interval points of the set $S_{\mathrm{sup}}$, and the segments on which the cocontrol varies are rather small. For small $h$, the number $l^*$ of components of the set $S_{\mathrm{sup}}$ weakly depends on $h$ and is (approximately) equal to the number of optimal control switching points in the corresponding problem without phase constraints. Therefore, for small $h$, interior points form the greater portion of elements of the set $S_{\mathrm{sup}}$. If the cumbersome routine (for small $h$) in correcting the initial support is eliminated, then the method is equivalent in complexity to the method applied in [4] for problems without phase constraints.

Moving (20) along the directions (23)–(25), we obtain

$$\begin{gathered}\alpha^1 = L - \zeta(s_* - h) < 0, \quad \alpha^{k+1} = \alpha^k + \Delta\alpha^k, \\ \Delta\alpha^k = 2Mh|\Delta\delta(t_l - h)| = 2Mh \quad \text{for} \quad \sigma^k = \sigma_v(t_l - h), \\ \Delta\alpha^k = 2L|\Delta\nu(s_l)| = 2L \quad \text{for} \quad \sigma^k = \sigma_u(s_l).\end{gathered}$$

For $\alpha^1 \leq \min\{2Mh, 2L\}$, we obtain $k_0 = 1$ (a short dual step is implemented). Otherwise, a long dual step would be implemented.

For $\sigma^* = \sigma_v(t_l - h)$, the new support is constructed by the rule

$$\overline{K}_{\mathrm{sup}} = \{\overline{S}_{\mathrm{sup}} = S_{\mathrm{sup}} \bigcup \{s_* - h\}, \quad \overline{T}_{\mathrm{sup}} = T_{\mathrm{sup}} \bigcup \{t_l - h\}\}.$$

For $\sigma^* = \sigma_u(s_l)$, we obtain

$$\overline{K}_{\mathrm{sup}} = \{\overline{S}_{\mathrm{sup}} = S_{\mathrm{sup}} \setminus s_l \bigcup \{s_* - h\}, \quad \overline{T}_{\mathrm{sup}} = T_{\mathrm{sup}}\},$$

i.e., a new support component (consisting of only one point $s_*$) is "generated," for which $\tau_l$ is the working support instant and the point $t_l$ becomes the working support instant for the component $\overline{S}_l$ of the set $\overline{S}_{\mathrm{sup}}$.

(2) $\left(s_* = s_l - h,\ \tau_l = t_l - h,\ l = \overline{2, l^*}\right)$. The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\mathrm{N}} \setminus s_*. \tag{26}$$

The corresponding variation of the cocontrol is

$$\begin{aligned}&\Delta\delta_h(t) = 1, \quad s^{l-1} \leq t \leq s_*, \\ &\Delta\delta_h(t) = 0, \quad t < s^{l-1}, \quad t > s_*,\end{aligned} \tag{27}$$

and the variation of the support component of the potential function is

$$\Delta\nu(s) = 0, \quad s \in S_{\mathrm{sup}}. \tag{28}$$

Moving (20) along directions (26)–(28), we obtain

$$\alpha^1 = L - \zeta(s_* - h) < 0 \quad (|L - \zeta(s_* - h)| \le Mh),$$
$$\alpha^{k+1} = \alpha^k + \Delta\alpha^k,$$
$$\Delta\alpha^k = 2Mh|\Delta\delta(\tau_*)| = 2Mh.$$

For small $h$, we have $\alpha^2 = \alpha^1 + \Delta\alpha^1 > 0$, i.e., a short dual step is realized. A new support is constructed by the rule

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}} \bigcup \{s_* - h\},\ \overline{T}_{\text{sup}} = T_{\text{sup}} \bigcup \tau_*\right\},$$

i.e., the working support instant for the component $\overline{S}_l$ of the set $\overline{S}_{\text{sup}}$ is displaced from the point $\tau_l = t_l - h$ to the point $\tau_*$.

(3) $(s_* = s_1 - h,\ \tau_1 \ne t_1 - h)$. The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\text{N}} \setminus s_*. \tag{29}$$

The corresponding variation of the cocontrol is

$$\Delta\delta_h(t) = 1, \quad t \le s_*, \quad \Delta\delta_h(t) = 0, \quad t > s_*, \tag{30}$$

and the variation of the component of the supporting component of the potential function is

$$\Delta\nu(s_1) = -1, \quad \Delta\nu(s) = 0, \quad s \in S_{\text{sup}} \setminus s_1. \tag{31}$$

Moving (20) along directions (29)–(31), we obtain

$$\alpha^1 = L - \zeta(s_* - h) < 0, \quad \alpha^{k+1} = \alpha^k + \Delta\alpha^k,$$
$$\Delta\alpha^k = 2Mh|\Delta\delta(\tau_*)| = 2Mh, \quad \text{for} \quad \sigma^k = \sigma_v(\tau_*),$$
$$\Delta\alpha^k = 2L|\Delta\nu(s_1)| = 2L, \quad \text{for} \quad \sigma^k = \sigma_u(s_1).$$

For $\alpha^1 \le \min\{2Mh, 2L\}$, we obtain $k_0 = 1$ (a short dual step is implemented). Otherwise, a long dual step would be implemented.

For $\sigma^* = \sigma_v(\tau_*)$, a new support is constructed by the rule

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}} \bigcup \{s_* - h\},\ \overline{T}_{\text{sup}} = T_{\text{sup}} \bigcup \tau_*\right\}.$$

For $\sigma^* = \sigma_u(s_1)$, we obtain

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = (S_{\text{sup}} \setminus s_1) \bigcup \{s_* - h\},\ \overline{T}_{\text{sup}} = T_{\text{sup}}\right\},$$

i.e., a new support component (consisting of only one point $s_*$) is "generated," for which $\tau_1$ is the working support instant, whereas $t_1$ becomes the working support instant for the component $\overline{S}_1$ of the set $\overline{S}_{\text{sup}}$.

(4) $(s_* = s_1 - h,\ \tau_1 = t_1 - h)$. The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\text{N}} \setminus s_*. \tag{32}$$

The corresponding variation of the cocontrol is

$$\Delta\delta_h(t) = 1, \quad t \le s_*, \quad \Delta\delta_h(t) = 0, \quad t > s_*, \tag{33}$$

and the variation of the support component of the potential function is

$$\Delta\nu(s) = 0, \quad s \in S_{\text{sup}}. \tag{34}$$

Moving (20) along directions (32)–(34), we obtain

$$\alpha^1 = L - \zeta(s_* - h) < 0 \quad (|L - \zeta(s_* - h)| \leq Mh),$$
$$\alpha^{k+1} = \alpha^k + \Delta\alpha^k,$$
$$\Delta\alpha^k = 2Mh|\Delta\delta(\tau_*)| = 2Mh.$$

For small $h$, we obtain $\alpha^2 = \alpha^1 + \Delta\alpha^1 > 0$, i.e., a short dual step is implemented. A new support is constructed by the rule

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}} \bigcup \{s_* - h\},\ \overline{T}_{\text{sup}} = T_{\text{sup}} \bigcup \tau_*\right\},$$

i.e., the working support instant for the component $\overline{S}_1$ of the set $\overline{S}_{\text{sup}}$ is displaced from the point $\tau_1 = t_1 - h$ to the point $\tau_*$.

(5) $\left(s_* = s^l + h,\ l = \overline{1, l^* - 1}\right)$. The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\text{N}} \setminus s_*. \tag{35}$$

The corresponding variation of the cocontrol is

$$\Delta\delta_h\left(t^l + h\right) = -1, \quad \Delta\delta_h(t) = 0, \quad t \in T_h \setminus \left\{t^l + h\right\}, \tag{36}$$

and the variation of the support component of the potential function is

$$\Delta\nu\left(s^l\right) = -1, \quad \Delta\nu(s) = 0, \quad s \in S_{\text{sup}} \setminus s^l. \tag{37}$$

Moving (20) along directions (35)–(37), we obtain

$$\alpha^1 = L - \zeta\left(s^l + h\right) < 0,\ \alpha^{k+1} = \alpha^k + \Delta\alpha^k,$$
$$\Delta\alpha^k = 2Mh|\Delta\delta\left(t^l + h\right)| = 2Mh,\ \sigma^k = \sigma_v\left(t^l + h\right),$$
$$\Delta\alpha^k = 2L|\Delta\nu\left(s^l\right)| = 2L,\ \sigma^k = \sigma_u\left(s^l\right).$$

For $\alpha^1 \leq \min\{2Mh, 2L\}$, we obtain $k_0 = 1$ (a short dual step is implemented). Otherwise, a long dual step would be implemented.

For $\sigma^* = \sigma_v\left(t^l + h\right)$, a new support is constructed by the rule

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}} \bigcup s_*,\ \overline{T}_{\text{sup}} = T_{\text{sup}} \bigcup \left\{t^l + h\right\}\right\}.$$

For $\sigma^* = \sigma_u\left(s^l\right)$, we obtain

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = \left(S_{\text{sup}} \setminus s^l\right) \bigcup s_*,\ \overline{T}_{\text{sup}} = T_{\text{sup}}\right\},$$

i.e., a new support component (consisting of only one point $s_*$) is "generated," for which $t^l$ is the working support instant.

(6) $(s_* = t^*)$. The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\mathrm{N}} \setminus s_*. \tag{38}$$

The corresponding variation of the cocontrol is

$$\Delta\delta_h(t) = -1, \quad s^{l^*} \le t < t^*, \quad \Delta\delta_h(t) = 0, \quad t < s^{l^*}, \tag{39}$$

and the variation of the support component of the potential function is

$$\Delta\nu(s) = 0, \quad s \in S_{\mathrm{sup}}. \tag{40}$$

Moving (20) along directions (38)–(40), we obtain

$$\begin{gathered}\alpha^1 = L - \zeta(t^*) < 0 \quad (|L - \zeta(t^*)| \le Mh),\\ \alpha^{k+1} = \alpha^k + \Delta\alpha^k,\\ \Delta\alpha^k = 2Mh|\Delta\delta(\tau_*)| = 2Mh.\end{gathered}$$

For small $h$, we obtain $\alpha^2 = \alpha^1 + \Delta\alpha^1 > 0$, i.e., a short dual step is implemented. A new support is constructed by the rule

$$\overline{K}_{\mathrm{sup}} = \left\{\overline{S}_{\mathrm{sup}} = S_{\mathrm{sup}} \bigcup s_*, \overline{T}_{\mathrm{sup}} = T_{\mathrm{sup}} \bigcup \tau_*\right\},$$

i.e., a new support component is "generated" at the point $t^*$.

(7) $(s_* < s_1,\ \tau_1 < s_*)$. The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\mathrm{N}} \setminus s_*. \tag{41}$$

The corresponding variation of the cocontrol is

$$\Delta\delta(t) = 1, \quad s_* \le t \le s_1 - h; \quad \Delta\delta(t) = 0, \quad t < s_*, \tag{42}$$

and the variation of the support component of the potential function is

$$\Delta\nu(s_1) = -1; \quad \Delta\nu(s) = 0, \quad s \in S_{\mathrm{sup}} \setminus s_1. \tag{43}$$

Moving (20) along directions (41)–(43), we obtain

$$\begin{gathered}\alpha^1 = L - \zeta(s_*) < 0 \quad (|L - \zeta(s_*)| \le M),\\ \Delta\alpha^k = 2M|\Delta\delta(\tau_*)| = 2M, \quad \text{for} \quad \sigma^k = \sigma_v(\tau_*),\\ \Delta\alpha^k = 2L|\Delta\nu(s_1)| = 2L, \quad \text{for} \quad \sigma^k = \sigma_u(s_1).\end{gathered}$$

For small $h$, we obtain $\alpha^2 = \alpha^1 + \Delta\alpha^1 > 0$, i.e., a short dual step is implemented. For $\sigma^* = \sigma_v(\tau_*)$, a new support is constructed by the rule

$$\overline{K}_{\mathrm{sup}} = \left\{\overline{S}_{\mathrm{sup}} = S_{\mathrm{sup}} \bigcup s_*,\ \overline{T}_{\mathrm{sup}} = T_{\mathrm{sup}} \bigcup \tau_*\right\},$$

i.e., a new component of the set $\overline{S}_{\mathrm{sup}}$ (consisting of only one point $s_*$) is "generated," for which $\tau_1$ is the working support instant, whereas $\tau_*$ becomes the working support instant for the component $\overline{S}_1$. For $\sigma^* = \sigma_u(s_1)$, we obtain

$$\overline{K}_{\mathrm{sup}} = \left\{\overline{S}_{\mathrm{sup}} = (S_{\mathrm{sup}} \setminus s_1) \bigcup s_*,\ \overline{T}_{\mathrm{sup}} = T_{\mathrm{sup}}\right\},$$

i.e., a component of the set $\overline{S}_{\text{sup}}$ is "generated," for which $\tau_1$ is the working support instant, whereas $t_1$ becomes the working support instant for the component $\overline{S}_1$.

(8) $(s_* < s_1,\ \tau_1 \geq s_*)$. The variation of the potential function is

$$\Delta\nu(s_*) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\text{N}} \setminus s_*. \tag{44}$$

The corresponding variation of the cocontrol is

$$\Delta\delta(t) = -1, \quad t \leq s_1 - h; \quad \Delta\delta(t) = 0, \quad t \geq s_*, \tag{45}$$

and the variation of the support component of the potential function is

$$\Delta\nu(s) = 0, \quad s \in S_{\text{sup}}. \tag{46}$$

Moving (20) along directions (44)–(46), we obtain

$$\alpha^1 = L - \zeta(s_*) < 0 \quad (|L - \zeta(s_*)| \leq Mh),$$
$$\Delta\alpha^k = 2Mh,$$
$$\alpha^2 = \alpha^1 + \Delta\alpha^1 > 0.$$

Consequently, a short dual step is implemented. A new support is constructed by the rule

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}} \bigcup s_*,\ \overline{T}_{\text{sup}} = T_{\text{sup}} \bigcup \tau_*\right\},$$

i.e., a new component of the set $\overline{S}_{\text{sup}}$ (consisting of only one point $s_*$) with working support instant $\tau_*$ is "generated."

After verifying the segment $[t_*, t^*]$, we go to step (III).

(III) We now eliminate the outliers $\omega(t)$, $t \in T_{\text{sup}}$, that lie outside the segment $[-M, M]$. Let $\tau \in T_{\text{sup}}$ denote the first instant on the left at which $|\omega(\tau)| > M$. Consider the case $\tau \in T^R_{\text{sup}}$.

For the sake of definiteness, let $\omega(\tau) > M$ (the case $\omega(\tau) < -M$ is investigated along similar lines). There are two possible situations: (1) $\tau \geq s_1$ and (2) $\tau < s_1$.

(1) $\tau \geq s_1$. The cocontrol variation is

$$\Delta\delta(\tau) = 1, \quad \Delta\delta(t) = 0, \quad t \in T_{\text{sup}} \setminus \tau. \tag{47}$$

The corresponding variations are

$$\Delta\nu(s_{l+1}) = -1, \quad \Delta\nu\left(s^l\right) = 1, \quad \Delta\nu(s) = 0, \quad s \in S_{\text{sup}} \setminus \left\{s_{l+1}, s^l\right\}, \tag{48}$$

$$\begin{aligned} &\Delta\delta(t) = 1, \quad t^l < t < t_{l+1}, \quad t \neq \tau, \\ &\Delta\delta(t) = 0, \quad t \leq t^l, \quad t \geq t^{l+1} + h, \quad t \in T_{\text{N}}. \end{aligned} \tag{49}$$

Moving (20) along directions (47)–(49), we find the initial rate $\alpha^1$ of decrement of the dual aim function is $M - \omega(\tau) < 0$; $\Delta\alpha^k = 2Mh$ if $\sigma^k = \sigma_v(t)$, $t \in \left[t^l + h, t_{l+1} - h\right] \setminus \tau$; $\Delta\alpha^k = 2L$ if $\sigma^k = \sigma_u(s)$, $s \in \left\{s^l, s_{l+1}\right\}$; and $\alpha^{k+1} = \alpha^k + \Delta\alpha^k$.

For $\alpha^1 \leq \min\{2Mh, 2L\}$, we obtain $k_0 = 1$ (a short dual step is implemented). Otherwise, a long dual step would be implemented.

For $\sigma^* = \sigma_v(\tau_*)$, a new support is constructed by the rule

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}},\ \overline{T}_{\text{sup}} = (T_{\text{sup}} \setminus \tau) \bigcup \tau_*\right\}.$$

Thus, the working support instant for the component $\overline{S}_{l+1}$ is displaced from the point $\tau$ to the point $\tau_*$. For $\sigma^* = \sigma_u(s_*)$, we have

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}} \setminus s_*,\ \overline{T}_{\text{sup}} = T_{\text{sup}} \setminus \tau\right\}.$$

As a result, the component $S_l$ (on the right) or $S_{l+1}$ (on the left) is eliminated, and the working support instant for the component $\overline{S}_{l+1}$ is displaced from the point $\tau$ to the point $s_*$.

(2) $\tau < s_1$. The cocontrol variation is

$$\Delta\delta(\tau) = 1, \quad \Delta\delta(t) = 0, \quad t \in T_{\text{sup}} \setminus \tau. \tag{50}$$

The corresponding variations are

$$\Delta\nu(s_1) = -1, \quad \Delta\nu(s) = 0, \quad s \in S_{\text{sup}} \setminus s_1, \tag{51}$$

$$\Delta\delta(t) = 1, \quad t < t_1, \quad t \neq \tau, \quad \Delta\delta(t) = 0, \quad t \geq t^1 + h, \quad t \in T_{\text{N}}. \tag{52}$$

Moving (20) along directions (50)–(52), we find that the initial rate $\alpha^1$ of decrement of the dual aim function is $M - \omega(\tau) < 0$; $\Delta\alpha^k = 2Mh$ if $\sigma^k = \sigma_v(t)$, $t \in \left[t^l + h, t_{l+1} - h\right] \setminus \tau$; $\Delta\alpha^k = 2L$ if $\sigma^k = \sigma_u(s_1)$; and $\alpha^{k+1} = \alpha^k + \Delta\alpha^k$.

For $\alpha^1 \leq \min\{2Mh, 2L\}$, we obtain $k_0 = 1$ (a short dual step is implemented). Otherwise, a long dual step would be implemented.

For $\sigma^* = \sigma_v(\tau_*)$, a new support is constructed by the rule

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}},\ \overline{T}_{\text{sup}} = T_{\text{sup}} \setminus \tau \bigcup \tau_*\right\}.$$

Thus, the working support moment for the component $\overline{S}_1$ is displaced from the point $\tau$ to the point $\tau_*$. For $\sigma^* = \sigma_u(s_1)$, we obtain

$$\overline{K}_{\text{sup}} = \left\{\overline{S}_{\text{sup}} = S_{\text{sup}} \setminus s_1,\ \overline{T}_{\text{sup}} = T_{\text{sup}} \setminus \tau\right\}.$$

Thus, the component $S_1$ is eliminated and its working support moment is displaced from the point $\tau$ to the point $s_1$.

After correcting such that the pseudosignal $\omega(t), t \in T_h$, does not escape the limits of $[-M, M]$, we verify whether the constraints on the pseudoaction $\zeta(s)$, $s \in S_h$ are violated or not. If they are violated, we go to step (II).

This method is finite under very broad general conditions. As in [3], we can design a finite modification of this method for any problem (1). The method terminates upon completion of the construction of an optimal support and its optimal accompanying pseudosignals and pseudoactions. If in the course of correction of an empty support, the increments of the dual aim function pile up (the initial value of the dual aim function on the empty support is known), then the optimal value of the dual aim function, which is equal to the optimal value of the quality criterion of problem (1), is generated at the end of this procedure.

Let us consider an example to illustrate the results of this section.

**Example 1.** Using an inertial control, an oscillator in equilibrium state at the initial instant $t_* = 0$ is to be excited to maximal rate in time $t^* = 7$. The model of the problem is

$$x_2(7) \to \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad \dot{u} = v, \tag{53}$$

$$x_1(0) = 0, \quad x_2(0) = 0, \quad u(0) = 0.5, \quad |u(t)| \leq 1, \quad |v(t)| \leq 1, \quad t \in T = [0, 7].$$

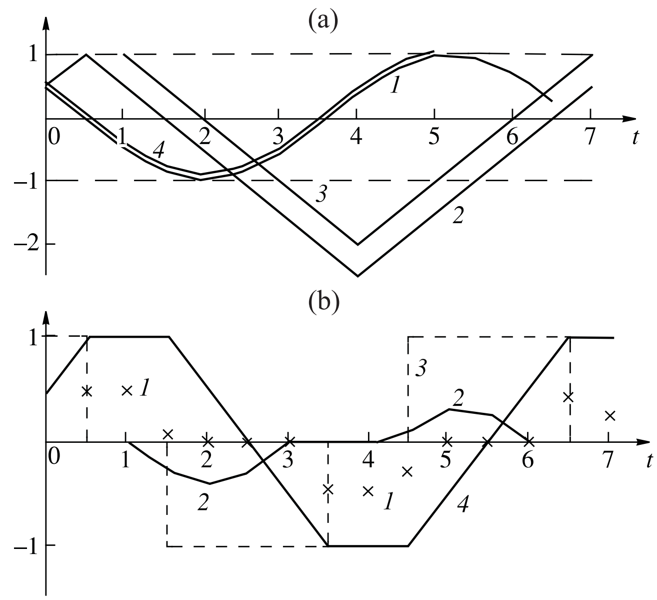


**Fig. 1.**

The quantization period is $h = 0.5$. Figure 1a shows the function $\psi^h_{n+1}(t)$, $t \in T_h$ (curve *1*) and pseudoaction $\zeta(s)$, $s \in S$ (curve *2*) accompanying the empty support. After eliminating the outlier $\zeta(5)$, we obtain the support

$$K_{\sup} = \{S_{\sup} = \{5\},\ T_{\sup} = \{0.5\}\},$$

which is accompanied by the pseudoaction $\zeta^1(s)$, $s \in S$ (curve *3*). The potential function $\nu(s)$, $s \in S_h$ (sequence of crosses 1), cocontrol $\Delta_h(t), t \in T_h$ (curve *2*), pseudosignal $\omega(t)$, $t \in T$ (curve *3*), and pseudoaction $\zeta(s)$, $s \in S$ (curve *4*) accompanying the optimal support are shown in Fig. 1b.

## 3. ATTAINABILITY SET

The attainability set $Q = Q(t^*) \subset R^n$ (at instant $t^*$) of the control system (1) is the set of all terminal states $x(t^* \mid t_*, x_0, u(\cdot))$ generated by the initial state $x(t_*) = x_0$ and all possible controls $u(\cdot) \in U^1$. For a control system, the attainability sets $Q(t^*), t^* \geq t_*$, play the same role as phase portraits in dynamic systems without any control.

In the general case, infinitely many parameters are required to describe the set $Q$ and, therefore, this set cannot be stored in the computer memory. Often approximations of the set $Q$ and its numerical characteristics (estimates) prove to be sufficient in applications.

Let a family $p_k \in R^n$, $k \in K$, $|K| > n$, of unit vectors densely fill a unit sphere. We assume than any $n$ vectors of this family are linearly independent and the elements $p_k$, $k \in K$, are ordered such that the adjacent $p_k$ and $p_{k+1}$ are close to one another. Let find the numbers $\alpha_k = \alpha_k(t^*) = p'_k x^k = \max p'_k x$, $x \in Q$, $k \in K$, that characterize the stretch of the set $Q$ along $p_k$, $k \in K$. The set $Q^* = Q^*(t^*) = \{x \in R^n : p'_k x \leq \alpha_k, k \in K\}$ is called the external approximation of the set $Q$. The internal approximation $Q_* = Q_*(t^*)$ of the set $Q$ is defined to be the convex shell of the points $x^k, k \in K$. Obviously, $Q_* \subset Q \subset Q^*$.

We construct the elements $\alpha_k$ and $x^k$, $k \in K$, defining the sets $Q^*$ and $Q_*$ by the following procedure. Using the method described in Section 2, we solve the optimal excitation problem (1), $c = p_1$. Thus we obtain $\alpha_1$ and $x^1$. Assume that problem (1) has been solved for $c = p_1, \ldots, p_k$, and the solution is $x_1, \ldots, x^k$; $\alpha_1, \ldots, \alpha_k$. The solution of problem (1) for $c = p_{k+1}$ is determined by correcting the solution of the previous problem (1), $c = p_k$.

Let $K^k_{\text{sup}} = \{S^k_{\text{sup}}, T^k_{\text{sup}}\}$ be the optimal support for problem (1), $c = p_k$. Let us determine $\psi^k_{n+1}(t)$, $t \in T$—the component of the solution of the adjoint system (4) under the initial conditions (5), $c = p_{k+1}$. Since $p_{k+1} = p_k + \Delta p_k$, we obtain $\psi^{k+1}_{n+1}(t) = \psi^k_{n+1}(t) + \Delta\psi^k_{n+1}(t)$, $t \in T$, where $\Delta\psi^k_{n+1}(t) = \Delta p'_k \int_t^{t^*} F(t^*)F^{-1}(\tau)b(\tau)d\tau$. Therefore, if the value $\overline{F}$ of the fundamental matrix $F(t)$, $t \in T$, at the point $\bar{t}$ is stored in the computer memory, we can quickly find the value $\psi^{k+1}_{n+1}(t)$, $t \in T$, in the neighborhood of the point $\bar{t}$ by integrating the equation $\dot{F} = A(t)F$, $F(\bar{t}) = \overline{F}$.

Using the functions $(\psi^h_{n+1})^{k+1}(t) = \int_t^{t+h} \psi^{k+1}_{n+1}(\tau)d\tau/h$, $t \in T_h$, let us construct the functions $\omega^{k+1}(t)$, $t \in T$, and $\zeta^{k+1}(s)$, $s \in S$, accompanying the support $K^k_{\text{sup}}$. According to the rules of construction of these functions (Section 2), the pseudoaction $\zeta^{k+1}(s)$, $s \in S$, satisfies the inequality $|\zeta^{k+1}(s)| \leq L$, $s \in S$, everywhere, but the pseudosignal $\omega^{k+1}(t)$, $t \in T$, may violate the constraint $|\omega^{k+1}(t)| \leq M$, $t \in T$ at the support instants $t \in T^k_{\text{sup}}$.

Let us correct the support $K^k_{\text{sup}}$ in order to eliminate the outliers $\omega^{k+1}(t)$, $t \in T^k_{\text{sup}}$, that lie outside the interval $[-M, M]$. Let $\tau \in T^k_{\text{sup}}$ be the first instant on the left at which $|\omega^{k+1}(\tau)| > M$. Let us consider the case $\tau \in (T^R_{\text{sup}})^k$.

For the sake of definiteness, we take $\omega^{k+1}(\tau) > M$. The cocontrol variation is

$$\Delta\delta(\tau) = 1, \quad \Delta\delta(t) = 0, \quad t \in T^k_{\text{sup}} \setminus \tau. \tag{54}$$

The corresponding variations are

$$\Delta\nu(s_{l+1}) = -1, \quad \Delta\nu\left(s^l\right) = 1, \quad \Delta\nu(s) = 0, \quad s \in S^k_{\text{sup}} \setminus \{s_{l+1}, s^l\}, \tag{55}$$

$$\begin{gathered} \Delta\delta(t) = 1, \quad t^l < t < t_{l+1}, \quad t \neq \tau, \\ \Delta\delta(t) = 0, \quad t \leq t^l, \quad t \geq t^{l+1} + h, \quad t \in T^k_{\text{N}}. \end{gathered} \tag{56}$$

Moving (20) along directions (54)–(56), we find that the initial rate $\alpha^1$ of decrement of the dual aim function is $M - \omega^{k+1}(\tau) < 0$; $\Delta\alpha^k = 2Mh$ if $\sigma^k = \sigma_v(t)$, $t \in \left[t^l + h, t_{l+1} - h\right] \setminus \tau$; $\Delta\alpha^k = 2L$ if $\sigma^k = \sigma_u(s)$, $s \in \left\{s^l, s_{l+1}\right\}$; and $\alpha^{k+1} = \alpha^k + \Delta\alpha^k$.

For $\alpha^1 \leq \min\{2Mh, 2L\}$, we obtain $k_0 = 1$ (a short dual step is implemented). Otherwise, a long dual step would be implemented.

For $\sigma^* = \sigma_v(\tau_*)$, a new support is constructed by the rule

$$K^{k+1}_{\text{sup}} = \left\{S^{k+1}_{\text{sup}} = S^k_{\text{sup}},\ T^{k+1}_{\text{sup}} = \{T^k_{\text{sup}} \setminus \tau\} \bigcup \tau_*\right\}.$$

Thus, the working support instant for the component $S^k_{l+1}$ is displaced from the point $\tau$ to the point $\tau_*$. For $\sigma^* = \sigma_u(s_*)$, we obtain

$$K^{k+1}_{\text{sup}} = \left\{S^{k+1}_{\text{sup}} = S^k_{\text{sup}} \setminus s_*,\ T^{k+1}_{\text{sup}} = T^k_{\text{sup}} \setminus \tau\right\}.$$

Hence the component $S^k_l$ (on the right) or $S^k_{l+1}$ (on the left) is eliminated, and the working support instant for the component $S^k_{l+1}$ is displaced from the point $\tau$ to the point $s_*$.

Operations involved in eliminating the outliers $\omega^{k+1}(t)$, $t \in T^k_{\text{sup}}$, for sufficiently small $\Delta p_k$, $k \in K$, admit a simple geometric interpretation. The optimality conditions (12) show that the components $S^k_l$, $l = \overline{1, l^*}$, of the set $S^k_{\text{sup}}$ correspond to segments of increment of the function $\psi^h_{n+1}(t)$, $t \in T_h$, if $\nu(s) \leq 0$, $s \in S^k_l$, and segments of decrement of the function $\psi^h_{n+1}(t)$, $t \in T_h$, if $\nu(s) > 0$, $s \in S^k_l$.

If there are no nonsupporting zeros of the cocontrol $\Delta_h(t)$, $t \in T_h$, between $S^k_l$ and $S^k_{l+1}$, then the working support instant $\tau_{l+1}$ lies at the point of nondecrement of the function $\psi^h_{n+1}(t)$, $t \in T_h$, if $\nu(s_{l+1}) > 0$, and at the point of nonincrement of this function if $\nu(s_{l+1}) < 0$.

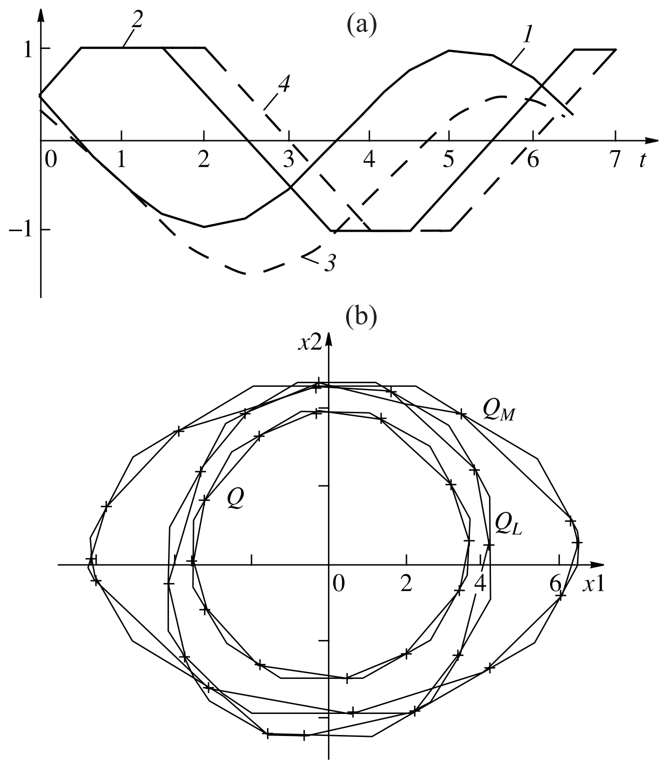


**Fig. 2.**

These properties of the optimal support are helpful in quickly eliminating the outliers $\omega^{k+1}(t)$, $t \in T^k_{\text{sup}}$, by displacing the point $\tau^k_{l+1}$ ($s^{lk} < \tau^k_{l+1} < s^k_{l+1} - h$) to the left or right (depending on the location of the points of maximum and minimum of the function $(\psi^h_{n+1})^{k+1}(t)$, $t \in T_h$), thereby eliminating or adding the components $S^k_l$, $S^k_{l+1}$ if $\tau^k_{l+1} = s^{lk}$ or $\tau^k_{l+1} = s^k_{l+1} - h$.

Let us consider an example illustrating the method of constructing an attainability set.

**Example 2.** Let us consider the oscillator of Example 1. Approximations of the attainability set were constructed with the vectors $p_k = (\cos(\pi(k-1)/6)$ and $\sin(\pi(k-1)/6))$, $k = \overline{1, 12}$. The point $x^1(t^*)$ was determined by solving Example 1. The pseudoaction accompanying the support $K^1_{\text{sup}}$ is shown in Fig. 2a (curve *2*), which also shows $(\psi^h_{n+1})^1(t)$, $t \in T_h$ (curve *1*), and $(\psi^h_{n+1})^2(t)$, $t \in T_h$ (curve *3*). The optimal support $K^2_{\text{sup}}$ can be constructed either with the algorithm described in this section or the properties of the optimal support stated above. The pseudoaction accompanying the support $K^2_{\text{sup}}$ is shown in Fig. 2a (curve *4*). It was used in finding the point $x^2(t^*) \in Q$. The points $x^3(t^*), \ldots, x^{12}(t^*)$ were determined along similar lines and used in finding the external $Q^*$ and internal $Q_*$ approximations of the attainability set (Fig. 2b). For the sake of comparison, Fig. 2 also shows the approximations of the attainability set for $L = \infty$ ($Q_M$) and $M = \infty$ ($Q_L$).

## 4. A LINEAR OPTIMAL CONTROL PROBLEM AND ITS OPEN-LOOP CONTROL

Let us apply the results of the optimal excitation problem for determining the solution of the following linear terminal control problem in the class $U^1$:

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \quad Hx(t^*) = g, \tag{57}$$
$$\dot{u} = v, \quad u(t_*) = u_0, \quad |u(t)| \le L, \quad |v(t)| \le M, \quad t \in T.$$

Here $H$ is an $m \times n$ matrix, $g$ is an $m$-vector, and $\operatorname{rank} H = m < n$. Let us determine the terminal set $X^* = \{x \in R^n : Hx = g\}$.

An inertial control $u(\cdot) \in U^1$ is said to be admissible for problem (57) if the trajectory $x(t)$, $t \in T$, of the dynamic system it generates reaches the terminal set $x(t^*) \in X^*$ at instant $t^*$.

The admissible control action $u^0(\cdot)$ and the corresponding control signal $v^0(\cdot)$ are said to be optimal if $c'x^0(t^*) = \max c'x(t^*)$, $u(\cdot) \in U^1$, on the (optimal) trajectory $x^0(t)$, $t \in T$, they generate.

For a given $\varepsilon \geq 0$, the suboptimal ($\varepsilon$-optimal) controls $u^\varepsilon(t)$, $v^\varepsilon(t)$, $t \in T$, and the trajectory $x^\varepsilon(t)$, $t \in T$, are defined by the inequality $c'x^0(t^*) - c'x^\varepsilon(t^*) \leq \varepsilon$.

Along with problem (57), let us consider the optimal excitation problem in the same class $U^1$:

$$c'(\lambda)x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \quad u(\cdot) \in U^1, \tag{58}$$

with the vector $c(\lambda) = c - H'\lambda(\lambda \in R^m)$.

Let $\lambda$ be an $m$-vector and $K_{\text{sup}}(\lambda) = \{S_{\text{sup}}(\lambda), T_{\text{sup}}(\lambda)\}$, the optimal support for problem (58).

According to the rule of Lagrange multipliers, for the optimal control $u^0(t)$, $t \in T$, of problem (57), there exists a vector $\lambda^0 \in R^m$ such that the solution $u^0(t|\lambda^0)$, $t \in T$, of problem (58) is $u^0(t)$, $t \in T$.

Combining this result with the maximum principle and criteria of optimality of control and support (Section 2), we obtain the following assertions for problem (58).

**The maximum principle.** For admissible control actions $u(\cdot)$ and signal $v(\cdot)$ to be optimal, it is necessary and sufficient that there exist a Lagrange vector $\lambda$ and a support $K_{\text{sup}}(\lambda) = \{S_{\text{sup}}(\lambda), T_{\text{sup}}(\lambda)\}$ such that (1) the condition of maximal action $\nu(s|\lambda)u(s) = \max\limits_{-L \leq u \leq L} \nu(s|\lambda)u$, $s \in S_{\text{sup}}(\lambda)$, and (2) the condition of maximal signal $\Delta_h(t|\lambda)v(t) = \max\limits_{-M \leq v \leq M} \Delta_h(t|\lambda)v$, $t \in T_{\text{N}}(\lambda)$, are satisfied for certain accompanying potential function $\nu(s|\lambda)$, $s \in S_h(\lambda)$, and cocontrol $\Delta_h(t|\lambda)$, $t \in T_h(\lambda)$.

**The $\boldsymbol{\varepsilon}$-maximum principle.** For any $\varepsilon \geq 0$, for an admissible control $u(\cdot)$ to be $\varepsilon$-optimal, it is necessary and sufficient that there exist a Lagrange vector $\lambda$ and a support $K_{\text{sup}}(\lambda) = \{S_{\text{sup}}(\lambda), T_{\text{sup}}(\lambda)\}$ such that (1) the condition of $\varepsilon$-maximal action $\nu(s|\lambda)u(s) = \max\limits_{-L \leq u \leq L} \nu(s|\lambda)u - \varepsilon_u(s)$, $s \in S_{\text{sup}}(\lambda)$, (2) the condition of $\varepsilon$-maximal signal $\Delta_h(t|\lambda)v(t) = \max\limits_{-M \leq v \leq M} \Delta_h(t|\lambda)v - \varepsilon_v(t)$, $t \in T_{\text{N}}(\lambda)$, and (3) the condition of $\varepsilon$-accuracy $\sum\limits_{s \in S_{\text{sup}}(\lambda)} \varepsilon_u(s) + \sum\limits_{t \in T_{\text{N}}(\lambda)} \varepsilon_v(t) \leq \varepsilon$ are satisfied for some accompanying potential function $\nu(s|\lambda)$, $s \in S_h(\lambda)$, and cocontrol $\Delta_h(t|\lambda)$, $t \in T_h(\lambda)$.

The Lagrange vector $\lambda^0$ and the support $K_{\text{sup}} = K_{\text{sup}}(\lambda^0)$ for which the maximum principle is satisfied are said to be optimal.

**Optimality criterion for the Lagrange vector and support**. For the Lagrange vector $\lambda$ and support $K_{\text{sup}}(\lambda)$ to be optimal, it is necessary and sufficient that $|\zeta(s|\lambda)| \leq L$, $s \in S_{\text{N}}(\lambda)$, and $|\omega(t|\lambda)| \leq M$, $t \in T_{\text{sup}}(\lambda)$, for certain accompanying pseudoaction and pseudosignal.

We now describe a method for constructing optimal open-loop controls. According to the duality theory, the optimal Lagrange vector $\lambda^0$ is the solution of the dual problem

$$\varphi_g(\lambda^0) = \min \varphi_g(\lambda), \quad \lambda \in R^m,$$

where

$$\varphi_g(\lambda) = \varphi(\lambda) + \lambda' g, \quad \varphi(\lambda) = \max c'(\lambda)x(t^*), \quad u(\cdot) \in U^1, \tag{59}$$

is the dual aim function. Function (59) is convex. We shall find its point of minimum in three steps: (1) construction of the initial approximation $\lambda^1$, (2) iterative refinement of the current value $\lambda^k$, and (3) final adjustment.

(1) Construction of $\lambda^1$. Using the method of [4], we solve two auxiliary problems in the class of discrete controls:

$(L)$ $\alpha_L = \max_u c'x(t^*)$, $\dot{x} = A(t)x + b(t)u$, $x(t_*) = x_0$, $x(t^*) \in X^*$, $|u(t)| \le L, t \in T$,

$(M)$ $\alpha_L = \max_v c'x(t^*)$, $\dot{x} = A(t)x + b(t)u$, $\dot{u} = v$, $x(t_*) = x_0$, $x(t^*) \in X^*$, $u(t_*) = u_0$, $|v(t)| \le M$, $t \in T$.

Let $\nu_L$ and $\nu_M$ be the optimal potential vectors of the problems $(L)$ and $(M)$, respectively. Taking $c_L = c - H'\nu_L$ and $c_M = c - H'\nu_M$, let us solve the optimal excitation problem (1) for $c = c_L$ and $c = c_M$. Let $\beta_L = (c'_L - \nu'_L H)x^0(t^*)$ and $\beta_M = (c'_M - \nu'_M H)x^0(t^*)$ be the optimal values of the quality criteria of these problems. Replacing $L$ and $M$ in problems $(L)$ and $(M)$ by $\overline{L} = L\beta_L/\alpha_L$ and $\overline{M} = M\beta_M/\alpha_M$, let us solve the problems $(\overline{L})$ and $(\overline{M})$. Let $\nu_{\overline{L}}$ and $\nu_{\overline{M}}$ be the optimal potential vectors of the new problems. Let us construct the accompanying $\zeta_{\overline{L}}(s)$, $\zeta_{\overline{M}}(s)$, $s \in S$, for problem (1) with $c = c_{\overline{L}}$ and $c = c_{\overline{M}}$. The corresponding trajectories of system $\dot{x} = A(t)x + b(t)u$, $x(t_*) = x_0$ are $x_{\overline{L}}(t)$, $x_{\overline{M}}(t)$, $t \in T$. Let us compute the mismatch vectors for the terminal constraints $Hx_{\overline{L}}(t^*) - g$ and $Hx_{\overline{M}}(t^*) - g$. Let $\lambda^1 = \nu_{\overline{L}}$ if $\|Hx_{\overline{L}}(t^*) - g\| \le \|Hx_{\overline{M}}(t^*) - g\|$ and $\lambda^1 = \nu_{\overline{M}}$ otherwise.

(2) Iterative improvement of the Lagrange vector. Using the initial approximation $\lambda^1$, let us construct subsequent approximations by the formula

$$\lambda^{k+1} = \lambda^k + \Theta_k(Hx^k(t^*) - g), \quad k = 1, 2, \ldots. \tag{60}$$

Here $\Theta_k \ge 0$ is a step that can be chosen by any gradient rule, $x^k(t^*)$ is the terminal state generated by the pseudoaction $\zeta(s|\lambda^k)$, $s \in S$, which accompanies the optimal support $K_{\sup}(\lambda^k)$ constructed in the optimal excitation problem (Section 2) for the vector $c = c(\lambda^k)$.

Iteration of (60) can be implemented quickly if the vector $x^k(t^*)$ yields to fast construction. To construct the vector $x^k(t^*)$ quickly using a known vector $x^{k-1}(t^*)$, let us express the latter by the Cauchy formula as the sum of a vector corresponding to the initial state $x_0$, $u_0$, and vectors, each of which corresponds to the segment of identical behavior of the pseudoaction $\zeta(s \mid \lambda^{k-1})$, $s \in S$ (time interval is called the segment of identical behavior if the pseudoaction is constant, or increases, or decreases in this interval). Under the substitution $\lambda^{k-1} \to \lambda^k$, these segments vary little and, therefore, the vectors corresponding to new segments can be constructed quickly if the states of the dynamic system at the endpoints of every segment are stored and the direct system is integrated on small intervals on which the old and new segments are different.

We compute the accompanying pseudoaction $\zeta(s|\lambda^{k+1})$, $s \in S$, by the method of Section 3. Taking $p_k = c - H'\lambda^k$ and $\Delta p_k = -H'(\lambda^{k+1} - \lambda^k) = \Theta_k(Hx^k(t^*) - g)$, we can reduce the construction of $\zeta(s \mid \lambda^{k+1})$, $s \in S$, to solving the optimal excitation problem for $p_{k+1}$ using the known solution for $p_k$. The method of solving such a problem was described in constructing attainability sets.

If $\varphi_g(\lambda^k) \to -\infty$ as $k \to \infty$ in the course of construction of sequential approximations (60), then the process of solving problem (57) is terminated, because there is no admissible control for this problem. In formula (60), we use only a few iterations for constructing a good approximation by the final adjustment, which is applied to solve problem (57) with the required accuracy.

(3) Final adjustment. Since control signals are discrete, the iterative procedure (2) when $\varphi(\lambda^k) \not\to -\infty$ as $k \to \infty$ leads us to a situation in which the mismatch of terminal constraints begin to fluctuate near values, whose magnitudes decrease with decreasing quantization period $h$. If iterations exhibit such a behavior, then we must pass to the final adjustment procedure.

Let $\lambda$ be the Lagrange vector on which procedure (2) was terminated and let $K_{\sup}(\lambda) = \{S_{\sup}(\lambda), T_{\sup}(\lambda)\}$ be the corresponding optimal support for the optimal excitation problem (1), $c = c(\lambda)$. "Freezing" the values $u(s) = L \operatorname{sgn} \nu(s)$, $s = s_l + h, \ldots, s^l - h$ and $v(t)$, $t \in T^* = \left\{t_l + h, \ldots, t^l - h\right\}$, $l = \overline{1, l^*}$, let us consider the optimal control problem with the intermediate phase constraints

$$
\begin{gathered}
c' x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \\
\dot{u} = v, \quad u(t_*) = u_0, \quad Hx(t^*) = g, \\
u(s_l - h)\operatorname{sgn}\nu(s_l) \le L, \quad u(s_l)\operatorname{sgn}\nu(s_l) \le L, \quad u(s_l + h) = L\operatorname{sgn}\nu(s_l), \\
u\left(s^l - h\right) = L\operatorname{sgn}\nu\left(s^l\right), \quad u\left(s^l\right)\operatorname{sgn}\nu\left(s^l\right) \le L, \\
u\left(s^l + h\right)\operatorname{sgn}\nu\left(s^l\right) \le L, \quad l = \overline{1, l^*}, \\
|v(t)| \le M, \quad t \in T \setminus T^*.
\end{gathered} \tag{61}
$$

A fast algorithm for solving problem (61) is given in [6]. There are two cases: (1) problem (61) has no solution and (2) the problem has a solution, but its structure requires the use of one more problem of the type (61). In case (1), instead of problem (61), we first eliminate the mismatch between terminal constraints [3]. In case (2), we correct the phase constraints and then solve the problem thus obtained. If the duality method [4] is applied, then every problem (61) yields to rapid solution and the solution of problem (61) is generated in a finite number of iterations.

Another method of final adjustment is described in the example given below. It is based on the following facts. In the general case, the optimal vector $\lambda^0$ is associated with a nonregular support $K_{\sup}(\lambda^0)$, which accompanies a set of pseudosignals and pseudoactions. Such a set is obtained if, for example, $\tau_{l+1} = s^l$, $\Delta_h(\tau_l \mid \lambda^0) = 0$, $\Delta_h(s_{l+1} \mid \lambda^0) = 0$, and $\nu(s_{l+1}|\lambda^0) = 0$. Choosing $\omega(\tau_{l+1})$, $\omega(s_{l+1})$, and $\zeta(s_{l+1})$, we obtain the pseudosignals and pseudoactions in the interval $\left[s^l, s_{l+1} + h\right]$ accompanying the support $K_{\sup}(\lambda^0)$ under direct constraints. For the terminal constraints $Hx(t^*) = g$, there exist $m$ intervals of the above type. If one accompanying pseudosignal is chosen on each of these intervals, then all $m$ terminal constraints are satisfied.

A method of constructing optimal open-loop controls is illustrated in the following example.

**Example 3.** An oscillator at instant $t^* = 7$ is to be transferred to the state $x_1(7) = 2$ at maximal speed. The mathematical model of the problem is

$$
\begin{gathered}
x_2(7) \to \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad \dot{u} = v, \quad x_1(0) = 0, \quad x_2(0) = 0, \\
x_1(7) = 2, \quad u(0) = 0.5, \quad |u(t)| \le 1, \quad |v(t)| \le 1, \quad t \in T = [0, 7].
\end{gathered} \tag{62}
$$

According the method described above, first we solve two auxiliary problems

$$
\begin{gathered}
(L) \quad \alpha_L = \max_u \sum_{t \in T_h} (-\sin(7 - (t+h)) + \sin(7-t))u(t), \\
\sum_{t \in T_h} (\cos(7 - (t+h)) - \cos(7-t))u(t) = 2, \quad |u(t)| \le 1, \quad t \in T_h;
\end{gathered} \tag{63}
$$

$$
\begin{gathered}
(M) \quad \alpha_L = \max_u \sum_{t \in T_h} (\cos(7 - (t+h)) - \cos(7-t))u(t), \\
\sum_{t \in T_h} (h + \sin(7 - (t+h)) - \sin(7-t))u(t) = 2 - (1 - \cos(7))0.5, \\
|u(t)| \le 1, \quad t \in T_h.
\end{gathered} \tag{64}
$$

Then we solve two optimal excitation problems

$$-\nu_L x_1(7) + x_2(7) \to \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad \dot{u} = v,$$
$$x_1(0) = 0, \quad x_2(0) = 0,$$
$$x_1(7) = 2, \quad u(0) = 0.5, \quad |u(t)| \le 1, \quad |v(t)| \le 1, \quad t \in T = [0, 7],$$
$$-\nu_M x_1(7) + x_2(7) \to \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad \dot{u} = v,$$
$$x_1(0) = 0, \quad x_2(0) = 0,$$
$$x_1(7) = 2, \quad u(0) = 0.5, \quad |u(t)| \le 1, \quad |v(t)| \le 1, \quad t \in T = [0, 7].$$

Let us solve the auxiliary problems (63) and (64), substituting $\overline{L}$ and $\overline{M}$ for $L$ and $M$, respectively. Then we solve two optimal excitation problems, substituting $\nu_{\overline{L}}$ and $\nu_{\overline{M}}$ for $\nu_L$ and $\nu_M$, respectively. Using the method described above, we compute the vectors of mismatches of terminal constraints and take $\lambda^1 = \nu_{\overline{L}} = -0.596$.

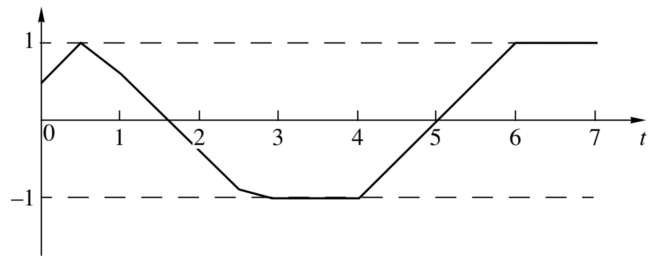


**Fig. 3.**

Using formula (60) with $\Theta = 0.01$ and $\lambda^1 = -0.596$, we construct the subsequent approximations. The pseudosignals and pseudoactions for the nonregular support $K_{\sup}(\lambda^0)$ are determined from the equation $\psi^h_{n+1}(0.5) = \psi^h_{n+1}(2.5)$. Hence $\lambda^0 = -0.59616$. Let us choose $\omega(0.5)$ and $\omega(2.5)$ using direct constraints such that the terminal constraint is satisfied. The pseudoaction for the optimal support is shown in Fig. 3.

## 5. OPTIMAL FEEDBACK CONTROL

First we find the optimal feedback control by imbedding problem (57) into the family of problems

$$\begin{gathered} c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(\tau) = y, \quad \dot{u} = v, \quad u(\tau) = z, \\ |u(s)| \le L, \quad s \in S(\tau) = [\tau, t^*], \quad |v(t)| \le M, \quad t \in T(\tau) = [\tau, t^*]. \end{gathered} \tag{65}$$

This family depends on the scalars $\tau \in T_h, z$ and $n$-vector $y$.

Let $v^0(t \mid \tau, y, z)$, $t \in T(\tau)$, be the optimal open-loop control signal of problem (65) for the position $(\tau, y, z)$ and let $G_\tau \subset R^{n+1}$ be the set of all states $(y, z)$ for which problem (65) has a solution. The function

$$v^0(\tau, y, z) = v^0(\tau \mid \tau, y, z), \quad (y, z) \in G_\tau, \quad \tau \in T_h, \tag{66}$$

is called the optimal (discrete) feedback control signal for problem (57).

For every nontrivial problem, it is not possible to find the optimal feedback (66) in explicit form. Therefore, as in [4], we describe a method of optimal feedback realization for every concrete control. The method is based on an analysis of the behavior of a control system with feedback (66).

Let us assume that the optimal feedback (66) has been constructed. Although it is designed from a deterministic model, it is obviously used to control a physical prototype of the model. Therefore, we assume that a real system is closed by feedback (66). Since a model always contains inaccuracies and unknown perturbations constantly act on the physical system in the course of control, let us write the equation of motion of a real physical optimal feedback system as

$$\dot{x} = A(t)x + b(t)u + w, \quad x(t_*) = x_0, \quad \dot{u} = v^0(t, x, u), \quad u(t_*) = u_0, \tag{67}$$

where $w$ is a set of terms describing the inaccuracies in modeling and perturbations acting on the object in the course of control. We refer to $w$ as perturbation. It is not known, but the exact values of $x^*(\tau)$ and $u^*(\tau)$ generated by the control $u^*(t)$, $t \in [t_*, \tau[$, and perturbation $w^*(t)$, $t \in [t_*, \tau]$, at the instant $\tau$ are accessible at every current instant $\tau \in T_h$ of the control process.

The trajectory $x(t), u(t), t \in T$, of the nonlinear system (67) with $x$- and $u$-discontinuous right side is defined to be the trajectory of the linear equations

$$\dot{x} = A(t)x + b(t)u + w, \quad x(t_*) = x_0, \quad \dot{u} = v(t), \quad u(t_*) = u_0,$$

with $v(t) = v(t_* + kh,\, x(t_* + kh),\, u(t_* + kh))$, $t \in [t_* + kh, t_* + (k+1)h[$, $k = \overline{0, N-1}$.

Let us consider a concrete control process for which the initial state is $x_0^*$, $u_0^*$, and perturbation is $w^*(t)$, $t \in T$. They generate in the closed-loop system (67) a process $x^*(t)$, $u^*(t)$, $t \in T$, satisfying the identities

$$\dot{x}^*(t) \equiv A(t)x^*(t) + b(t)u^*(t) + w^*(t), \quad \dot{u}^*(t) \equiv v^0(t, x^*(t), u^*(t)), \quad t \in T. \tag{68}$$

By (68), it is clear the signal

$$v^*(t) = v^0(t, x^*(t), \quad u^*(t)), \quad t \in T, \tag{69}$$

is fed to the input of system (67) in the course of control. This signal is called the realization of the optimal feedback (66) in the control process.

According to (69), there is no need to know the optimal feedback (66) as a whole (in the whole domain of its definition) to control a concrete process, it suffices to know only its values along certain isolated continuous curves $x^*(t)$, $u^*(t)$, $t \in T$. Furthermore, it suffices to compute at every current instant $\tau \in T_h$ the current value of $v^*(\tau) = v^0(\tau, x^*(\tau), u^*(\tau))$ from the measurements on $x^*(\tau)$ and $u^*(\tau)$ in a time interval not greater than $h$, i.e., in real time scale. As in [4], we refer to the device that implements such an operation as the optimal controller.

Thus, the design of an optimal feedback (66) is reduced to constructing an algorithm for the operation of the optimal controller.

**Algorithm for the operation of the optimal controller.** In the initial interval $[t_*, t_* + h]$, in which the initial state $x_0, u_0$ is known, the optimal controller is virtually idle. It is called upon just to feed system (57) with the signal $v^*(t) = v^0(t_*, x_0, u_0) = v^0(t_*|t_*, x_0, u_0)$, $t \in [t_*, t_* + h[$ determined from the optimal program $v^0(t|t_*, x_0, u_0)$, $t \in T$, which is constructed in advance prior to the commencement of the control process without any constraints on computation time.

Let us study the situation encountered by the optimal controller at a current instant $\tau = t_* + kh \in T_h$ of the control process. By assumption, the controller at the preceding instant $\tau - h$ must have generated the signal $v^*(\tau - h) = v^0(\tau - h|\tau - h,\ x^*(\tau - h),\ u^*(\tau - h))$ and knows the optimal open-loop control signal $v^0(t\,|\,\tau - h,\ x^*(\tau - h),\ u^*(\tau - h))$, $t \in [\tau - h, t^*]$, optimal Lagrange vector $\lambda^0(\tau - h)$, and optimal support $K_{\text{sup}}(\tau - h \mid \lambda^0(\tau - h))$. Let $x^0(\tau)$, $u^0(\tau)$, denote the state to which the system would have passed at instant $\tau$ from the state $x^*(\tau - h)$, $u^*(\tau - h)$ without perturbations. The real state $x^*(\tau)$, $u^*(\tau)$ differs from the ideal state $x^0(\tau)$, $u^0(\tau)$, by the vector

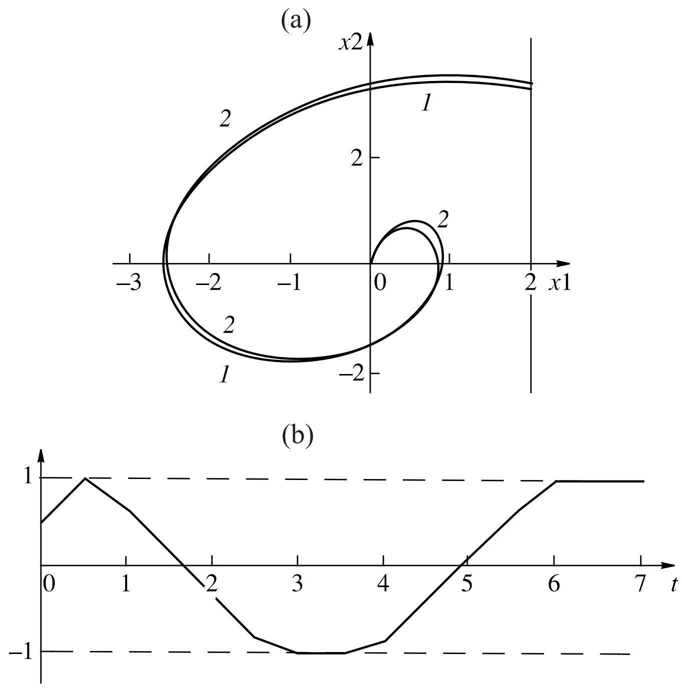


**Fig. 4.**

$\int_{\tau-h}^{\tau} F(\tau)F^{-1}(s)w^*(s)ds$. Under bounded perturbation, the lesser the magnitude of $h$, the lesser the value of $\left\|\int_{\tau-h}^{\tau} F(\tau)F^{-1}(s)w^*(s)ds\right\|$. Therefore, the optimal controller applies the dual method to correct $\lambda^0(\tau-h)$ and $K_{\text{sup}}(\tau-h|\lambda^0(\tau-h))$ until the optimal vector $\lambda^0(\tau)$ and optimal support $K_{\text{sup}}(\tau|\lambda^0(\tau))$ are constructed. This correction can be implemented either by procedures (2) and (3) of Section 4, or by the final adjustment procedure, taking the instant $\tau$, vector $\lambda^0(\tau-h)$, support $K_{\text{sup}}(\tau-h \mid \lambda^0(\tau-h))$, and the state $x^*(\tau), u^*(\tau)$ as the initial values.

As is clear from the description of the algorithm for the optimal controller operation, the method of realization of optimal feedback in the class of inertial controls for problem (1) is as effective as the similar method for the optimal control problem in the class of discrete controls [4]. The usefulness of the latter can be judged by the results of numerical experiments reported in [4].

The following example illustrates our optimal feedback designing method.

**Example 4.** Let us solve the problem of Example 3 of the previous section. Figure 4a shows the trajectory of an optimal closed-loop system without perturbations (curve *1*) and the trajectory of this system under the action of the perturbation $w^*(t) = 0.2\sin(2t)$, $t \in [0,4]$, $w^*(t) = 0$, $t \in ]4,7]$ (curve *2*). The optimal feedback realization $v^*(t)$, $t \in T$, is shown in Fig. 4b.

## 6. CONCLUSIONS

If the specifics of phase constraints are adequately taken into account, then it is possible to design effective methods of computing optimal programs and optimal feedback controls for non-trivial optimal control problems in which not only values, but also derivatives of control actions are constrained. The method is based, not on the well-known maximum principle and dynamic programming procedure, but on a far simpler linear programming technique. This paper is one more example (besides the classical example of transport problems) to illustrate the effectiveness of an approach in which a realization of the general method is designed for a special problem rather than solving the problem by general methods.

*APPENDIX*

(1) The Cauchy formula [7] for the extended dynamic system (2) takes the form

$$\overline{x}(t) = \overline{F}(t)\overline{F}^{-1}(t_*)\overline{x}_0 + \int\limits_{t_*}^{t} \overline{F}(t)\overline{F}^{-1}(\tau)\overline{b}(\tau)v(\tau)d\tau,$$

where $\overline{F}(t)$, $t \in T$, is the fundamental $(n+1)\times(n+1)$ matrix of solutions of the extended system (2)

$$\dot{\overline{F}} = \overline{A}(t)\overline{F}, \quad \overline{F}(t_*) = \overline{E}. \tag{A.1}$$

Let us write Eq. (A.1) in block form as

$$\begin{pmatrix} \dot{F} & \dot{f} \\ \dot{g}' & \dot{h} \end{pmatrix} = \begin{pmatrix} A(t) & b(t) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F & f \\ g' & h \end{pmatrix}.$$

Block multiplying the matrices on the right, we obtain

$$\dot{F} = A(t)F, \quad F(t_*) = E; \quad \dot{f} = A(t)f + b(t), \quad f(t_*) = 0; \quad g(t) \equiv 0;$$
$$h(t) \equiv 1, \quad t \in T.$$

Hence $\overline{F}(t) = \begin{pmatrix} F(t) & f(t) \\ 0 & 1 \end{pmatrix}$, $f(t) = \int_{t_*}^{t} F(t)F^{-1}(\tau)b(\tau)d\tau$.

The inverse $\overline{F}^{-1}(t)$ of the matrix $\overline{F}(t)$ is given by the equation $\overline{F}(t)\overline{F}^{-1}(t) = \overline{E}$:

$$\overline{F}^{-1}(t) = \begin{pmatrix} F^{-1}(t) & -F^{-1}(t)f(t) \\ 0 & 1 \end{pmatrix}.$$

The adjoint of the extended dynamic system (2) is

$$\dot{\overline{\psi}}' = (\dot{\psi}, \dot{\psi}_{n+1})' = -(\psi, \psi_{n+1})' \begin{pmatrix} A(t) & b(t) \\ 0 & 0 \end{pmatrix} = -(\psi' A(t), \psi' b(t)),$$
$$(\psi(t^*), \psi_{n+1}(t^*)) = (c, 0).$$

Hence

$$\dot{\psi}' = -\psi' A(t), \quad \dot{\psi}_{n+1} = -\psi' b(t), \quad \psi(t^*) = c, \quad \psi_{n+1}(t^*) = 0;$$
$$\psi'(t) = c' F(t^*)F^{-1}(t), \quad \psi_{n+1}(t) = -\int\limits_{t^*}^{t} c' F(t^*)F^{-1}(\tau)b(\tau)d\tau.$$

Applying these formulas, we obtain

$$\overline{c}'\overline{x}(t^*) = c'x(t^*) = \overline{c}'\overline{F}(t^*)\overline{F}^{-1}(t_*) \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} + \int\limits_{t_*}^{t^*} \overline{c}'\overline{F}(t^*)\overline{F}^{-1}(\tau)\overline{b}(\tau)v(\tau)d\tau$$
$$= c' F(t^*)F^{-1}(t_*)x_0 + (c' f(t^*) - c' F(t^*)F^{-1}(t_*)f(t_*))u_0 + \int\limits_{t_*}^{t^*} \overline{\psi}'(\tau)\overline{b}(\tau)v(\tau)d\tau$$
$$= \sum_{t \in T_h} v(t) \int\limits_{t}^{t+h} \psi_{n+1}(\tau)d\tau + c' F(t^*)F^{-1}(t_*)x_0 + (c' f(t^*) - c' F(t^*)F^{-1}(t_*)f(t_*))u_0$$
$$= c' F(t^*)F^{-1}(t_*)x_0 + (c' f(t^*) - c' F(t^*)F^{-1}(t_*)f(t_*))u_0 + \sum_{t \in T_h} c(t)v(t).$$

(2) The interval linear programming problem [3] takes the form

$$c'x \to \max, \quad b_* \leq Ax \leq b^*, \quad d_* \leq x \leq d^*, \tag{A.2}$$

where $A = A(I, J) = (a_{ij},\ i \in I = \{1, 2, \ldots, m\},\ j \in J = \{1, 2, \ldots, n\})$.

If all constraints of problem (A.2) are satisfied, then $x = x(J)$ (vectors) and $z = z(I) = Ax$ (plan and cost vector). The plan $x^0 \in X$ is optimal if $c'x^0 = \max c'x$, $x \in X$, where $X$ is the plan set. For a given $\varepsilon \geq 0$, the suboptimal ($\varepsilon$-optimal) plan $x^\varepsilon$ is $c'x^0 - c'x^\varepsilon \leq \varepsilon$, $x^\varepsilon \in X$.

The set $K_{\text{sup}} = \{I_{\text{sup}}, J_{\text{sup}}\}$, $I_{\text{sup}} \subset I$, $J_{\text{sup}} \subset J$, $|I_{\text{sup}}| = |J_{\text{sup}}|$, is called the support if $\det A_{\text{sup}} \neq 0$, where $A_{\text{sup}} = A(I_{\text{sup}}, J_{\text{sup}})$. By definition, the empty set $K_{\text{sup}} = \{I_{\text{sup}} = \varnothing, J_{\text{sup}} = \varnothing\} = \varnothing$ is a support.

The support is accompanied by (1) the potential vector (the Lagrange vector) $\nu = (\nu_{\text{sup}} = (\nu_i, i \in I_{\text{sup}}), \nu_{\text{N}} = (\nu_i, i \in I_{\text{N}}) = 0)$, $\nu'_{\text{sup}} = c'_{\text{sup}} A^{-1}_{\text{sup}}$, $\nu_{\text{sup}} = \nu(I_{\text{sup}})$, $c_{\text{sup}} = c(J_{\text{sup}})$, $I_{\text{N}} = I \setminus I_{\text{sup}}$, (2) the estimate vector $\Delta = (\Delta_{\text{sup}} = (\Delta_j, j \in J_{\text{sup}}) = 0, \Delta_{\text{N}} = (\Delta_j, j \in J_{\text{N}}))$, $\Delta'_{\text{N}} = c'_{\text{N}} - \nu'_{\text{sup}} A(I_{\text{sup}}, J_{\text{N}})$, $J_{\text{N}} = J \setminus J_{\text{sup}}$, (3) the pseudoplan $\varkappa = \varkappa(J)$, and the pseudocost vector $\xi = A\varkappa$:

$$\xi_i = b_{*i} \quad \text{if} \quad \nu_i < 0; \qquad \xi_i = b^*_i \quad \text{if} \quad \nu_i > 0;$$
$$\xi_i \in [b_{*i}, b^*_i] \quad \text{if} \quad \nu_i = 0, \qquad i \in I_{\text{sup}},$$
$$\varkappa_j = d_{*j} \quad \text{if} \quad \Delta_j < 0,$$
$$\varkappa_j = d^*_j \quad \text{if} \quad \Delta_j > 0, \qquad \varkappa_j \in [d_{*j}, d^*_j] \quad \text{if} \quad \Delta_j = 0, \qquad j \in J_{\text{N}},$$
$$\varkappa_{\text{sup}} = \varkappa(J_{\text{sup}}) = A^{-1}_{\text{sup}}(\xi_{\text{sup}} - A(I_{\text{sup}}, \quad J_{\text{N}})\varkappa_{\text{N}}), \quad \xi_{\text{N}} = \xi(I_{\text{N}}) = A(I_{\text{N}}, J)\varkappa.$$

A support is said to be regular if $\nu_i \neq 0$, $i \in I_{\text{sup}}$, $\Delta_j \neq 0$, $j \in J_{\text{N}}$. A regular support is accompanied by unit potential and estimate vectors.

If $K_{\text{sup}}$ is an empty support, then $\nu = 0$, $\Delta = \Delta_{\text{N}}$, $\varkappa = \varkappa_{\text{N}}$, $\xi = \xi_{\text{N}}$.

A pair $\{x, K_{\text{sup}}\}$ consisting of a plan and a support is called the support plan.

**Theorem A.1** (optimality criterion). *For a plan $x$ to be optimal, it is necessary and sufficient that there exist a support $K_{\text{sup}}$ for which the support plan $\{x, K_{\text{sup}}\}$ satisfies the maximum condition*

$$\nu_i z_i = \max_{b_{*i} \leq \xi \leq b^*_i} \nu_i \xi, \quad i \in I_{\text{sup}}; \quad \Delta_j x_j = \max_{d_{*j} \leq \omega \leq d^*_j} \Delta_j \omega, \quad j \in J_{\text{N}}.$$

*A support is said to be optimal if it satisfies the optimality condition.*

**Theorem A.2** (suboptimality criterion). *For any $\varepsilon \geq 0$, a plan $x$ is $\varepsilon$-optimal if and only if there exist a support $K_{\text{sup}}$ such that the plan $\{x, K_{\text{sup}}\}$ satisfies the $\varepsilon$-maximum condition*

$$\nu_i z_i = \max_{b_{*i} \leq \xi \leq b^*_i} \nu_i \xi - \varepsilon_{zi}, \quad i \in I_{\text{sup}};$$
$$\Delta_j x_j = \max_{d_{*j} \leq \omega \leq d^*_j} \Delta_j \omega - \varepsilon_{xj}, \quad j \in J_{\text{N}}; \quad \sum_{i \in I_{\text{sup}}} \varepsilon_{zi} + \sum_{j \in J_{\text{N}}} \varepsilon_{xj} \leq \varepsilon.$$

**Theorem A.3** (support optimality criterion). *For a support $K_{\text{sup}}$ to be optimal, it is necessary and sufficient that the relations*

$$d_{*j} \leq \varkappa_j \leq d^*_j, \quad j \in J_{\text{sup}}; \quad b_{*i} \leq \xi_i \leq b^*_i, \quad i \in I_{\text{N}},$$

*hold for certain accompanying pseudoplan and pseudocost vector.*

*If these inequalities hold, then $\varkappa$ and $\xi$ are optimal plan and optimal cost vector, respectively.*

The dual method. Let $K_{\text{sup}} = \{I_{\text{sup}}, J_{\text{sup}}\}$ be an initial support (it can also be an empty support) and let $\nu = (\nu(I_{\text{sup}}), \nu(I_{\text{N}}) = 0)$, $\Delta = (\Delta(J_{\text{sup}}) = 0, \Delta(J\text{N}))$, $\varkappa$, $\xi$ be the accompanying potential vector, estimate vector, pseudoplan, and pseudocost vector ($\nu = 0$ if $K_{\text{sup}} = \varnothing$), respectively.

Let $I_{\text{N}}^- = \{i \in I_{\text{N}} : \xi_i \notin [b_{*i}, b_i^*]\}$, $J_{\text{sup}}^- = \{j \in J_{\text{sup}} : \varkappa_j \notin [d_{*j}, d_j^*]\}$, and $K^- = I_{\text{N}}^- \bigcup J_{\text{sup}}^-$. If $K^- = \varnothing$, then the support $K_{\text{sup}}$ and the plan $\varkappa = x^0$ are optimal.

For $K^- \neq \varnothing$, let us take an arbitrary element from $K^-$. Here two cases are possible: the element belongs either to the set $I_{\text{N}}^- : i_0 \in I_{\text{N}}^-$ or to the set $J_{\text{sup}}^- : j_0 \in J_{\text{sup}}^-$.

(1) ($i_0 \in I_{\text{N}}^-$). Let us start with the construction of $\Delta y_{\text{N}} = (\Delta y_i,\, i \in I_{\text{N}})$:

$$\Delta y_{i0} = \begin{cases} \phantom{-}1 & \text{if } \xi_{i0} > b_{i0}^* \\ -1 & \text{if } \xi_{i0} < b_{*i_0}, \end{cases} \qquad \Delta y_i = 0, i \in I_{\text{N}} \setminus i_0. \tag{A.3}$$

Let us construct the component $\Delta y_{\text{sup}} = (\Delta y_i, i \in I_{\text{sup}})$ of the vector $\Delta y = (\Delta y_{\text{sup}}, \Delta y_{\text{N}})$, $K_{\text{sup}} \neq \varnothing$:

$$\Delta y'_{\text{sup}} = -\Delta y_{i_0} A(i_0, J_{\text{sup}})' A_{\text{sup}}^{-1}. \tag{A.4}$$

If $K_{\text{sup}} = \varnothing$, then we take $\Delta y_{\text{sup}} = 0$.

The variation corresponding to the vector $\Delta y = (\Delta y_{\text{sup}} \Delta y_{\text{N}})$ is

$$\Delta\delta' = (\Delta\delta'_{\text{N}} = -\Delta y' A(I, J_{\text{N}}); \Delta\delta'_{\text{sup}} = 0). \tag{A.5}$$

When

$$y(\sigma) = \nu + \sigma\Delta y, \quad \delta(\sigma) = \Delta + \sigma\Delta\delta, \quad \sigma \geq 0 \tag{A.6}$$

moves from the point $(\nu, \Delta)$ along variations (A.3)–(A.5), the initial rate of decrement of the aim function is $\alpha^1 = b_{i0}^* - \xi_{i0}$ if $\xi_{i0} > b_{i0}^*$ and $\alpha^1 = \xi_{i0} - b_{*i0}$ if $\xi_{i0} < b_{*i0}$.

Let us find the zeros of the function (A.6)

$$\begin{aligned} \sigma_i &= \begin{cases} -\nu_i/\Delta y_i & \text{if } \nu_i \Delta y_i < 0 \\ \infty & \text{if } \nu_i \Delta y_i \geq 0, \quad i \in I_{\text{sup}}, \end{cases} \\ \sigma_j &= \begin{cases} -\Delta_j/\Delta\delta_j & \text{if } \Delta_j \Delta\delta_j < 0 \\ \infty & \text{if } \Delta_j \Delta\delta_j \geq 0, \quad j \in J_{\text{N}}. \end{cases} \end{aligned} \tag{A.7}$$

For $K_{\text{sup}} = \varnothing$, we take $\sigma_i = \infty$, $i \in I_{\text{sup}}$.

Let us order the finite numbers in (A.7) in increasing order of magnitude: $\sigma^1 < \sigma^2 < \ldots < \sigma^{k*}$ (if there are no such numbers, then the constraints of problem (A.2) are inconsistent). The dual aim function is linear relative to $\sigma$ between the points $\sigma^k$ and $\sigma^{k+1}$ and its rate $\alpha^k$ has a jump at the point $\sigma^{k+1}$

$$\Delta\alpha^k = \sum_{i \in I^k} (b_i^* - b_{*i})|\Delta y_i| + \sum_{j \in J^k} (d_j^* - d_{*j})|\Delta\delta_j|,$$
$$I^k = \{i \in I_{\text{sup}} : y_i(\sigma^k) = 0\}, \quad J^k = \{j \in J_{\text{N}} : \delta_j(\sigma^k) = 0\}.$$

Hence $\alpha^{k+1} = \alpha^k + \Delta\alpha^k$.

Let $k_0$ be a superscript for which $\alpha^{k_0} < 0$, $\alpha^{k_0+1} \geq 0$. The number $\sigma^* = \sigma^{k_0}$ is called the short dual step if $k_0 = 1$, otherwise $\sigma^*$ is said to be a long dual step. Let choose any element from the set $I^{k_0} \cup J^{k_0}$. Here two cases are possible: (1a) the element belongs to set $I^{k_0} : i_* \in I^{k_0}$ or

$(1b)$ to the set $J^{k_0} : j_* \in J^{k_0}$. Let us form the components of the new support $\overline{K}_{\text{sup}} = \{\overline{I}_{\text{sup}}, \overline{J}_{\text{sup}}\}$: $(1a)$ $\overline{I}_{\text{sup}} = (I_{\text{sup}} \setminus i_*) \cup i_0$, $\overline{J}_{\text{sup}} = J_{\text{sup}}$; $(1b)$ $\overline{I}_{\text{sup}} = I_{\text{sup}} \cup i_0$, $\overline{J}_{\text{sup}} = J_{\text{sup}} \cup j_*$.

(2) $(j_0 \in J^-_{\text{sup}})$. Let us begin with the construction of $\Delta\delta_{\text{sup}} = (\Delta\delta_j,\, j \in J_{\text{sup}})$:

$$\Delta\delta_{j0} = \begin{cases} 1 & \text{if} \quad \varkappa_{j0} > d^*_{j0} \\ -1 & \text{if} \quad \varkappa_{j0} < d_{*j_0}, \end{cases} \qquad \Delta\delta_j = 0, j \in J_{\text{sup}} \setminus j_0. \tag{A.8}$$

The variations corresponding to the vector $\Delta\delta_{\text{sup}}$ are

$$\Delta y'_{\text{sup}} = -e'_{j_0} A^{-1}_{\text{sup}} \Delta\delta_{j_0}, \quad \Delta y_{\text{N}} = 0, \tag{A.9}$$
$$\Delta\delta'_{\text{N}} = -\Delta y'_{\text{sup}} A(I_{\text{sup}}, J_{\text{N}}), \ (e_{j_0} \text{ is a unit vector}).$$

When (A.6) moves from $(\nu, \Delta)$ along (A.8) and (A.9), the initial rate of the dual aim function is

$$\alpha^1 = d^*_{j0} - \varkappa_{j0} \quad \text{if} \quad \varkappa_{j0} > d^*_{j0},$$
$$\alpha^1 = \varkappa_{j0} - d_{*j0} \quad \text{if} \quad \varkappa_{j0} < d_{*j0}.$$

Then we investigate along the same lines as in case (1).

The new support is of the form $(2a)$ $\overline{K}_{\text{sup}} = \{\overline{I}_{\text{sup}}, \overline{J}_{\text{sup}}\}$, $\overline{I}_{\text{sup}} = I_{\text{sup}} \setminus i_*$, $\overline{J}_{\text{sup}} = J_{\text{sup}} \setminus j_0$, $(2b)$ $\overline{K}_{\text{sup}} = \{\overline{I}_{\text{sup}}, \overline{J}_{\text{sup}}\}$, $\overline{I}_{\text{sup}} = I_{\text{sup}}$, $\overline{J}_{\text{sup}} = (J_{\text{sup}} \setminus j_0) \cup j_*$. In every case, the support $\overline{K}_{\text{sup}}$ is accompanied by a potential vector $\overline{y} = \nu + \sigma^* \Delta y$ and an estimate vector $\overline{\delta} = \Delta + \sigma^* \Delta\delta$. The method is finite under very broad general conditions [3]. Finite modifications of the method are described in [3].

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