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DETERMINATE SYSTEMS

Optimal Control by Dynamic Controllers[1]

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Abstract—Problems of optimal indirect control are considered. It is assumed that information on the behavior of a controllable object first arrives at a control element that generates control signals for a dynamic controller. Using the external energy, the dynamic controller produces control actions of required power that enter the input of the controllable object. Control signals and actions are bounded by geometric constraints. The aim of control is to transfer an object to a prescribed terminal set with the security of a maximum value of the terminal quality criterion. The problems under consideration, which contain phase constraints, are solved by means of the earlier developed methods of solving the problems of optimal control of dynamic objects with simpler intermediate phase constraints. In this case, the main attention is given to the use of these methods for the effective implementation in real time of positional solutions (optimal controls of the feedback type). Examples are given to illustrate the obtained results.

1. INTRODUCTION

The classical mathematical theory of optimal processes [1] relies on the mathematical models of direct control systems. In this case, low-power control actions are formed directly from signals of the devices that measure output signals of a controllable object. Most of the real control systems are indirect control ones. In these systems, information on the behavior of a controllable object first arrives at a control element that works out control signals for a controller on the basis of this information. Using external energy sources, the controller produces control actions of required power, which are fed to the controllable object. The formal unification of the models of a controllable object and a dynamic controller does not make it possible to use effectively the available classical results because in this case the problems of optimal control (OC) with phase constraints appear, for which good-quality effective solution methods do not yet exist. The mathematical investigations [2] even of simple OC problems with phase constraints reveal that exotic modes often arise in these problems, which were first found by Fuller [3] in linear-quadratic OC problems.

In the literature, works are rather often found in which applied OC problems with phase constraints are solved by the mathematical programming method. In this case, a problem is digitized completely and solved by any numerical method of mathematical programming. The critical review of this approach is presented in [4]. From our viewpoint, the basic disadvantage of the method of mathematical programming consists in the fact that it is impossible to use this method for a sufficiently exact implementation of the feedback-type OC in specific control problems when constraints on the size of the on-line storage and the storage speed play an essential role.

In this work, we do not examine the issue of the effectiveness of the suggested method of the solution of a stated problem. In accordance with [5], it can be estimated for each particular problem. The effectiveness will be high for certain because all operations of the outlined method rest on the method of the solution of an auxiliary problem, the effectiveness of which was already checked. As in [5], here the dual method of the solution of OC problems plays a crucial role.

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The aim of this work is to outline methods of the solution of OC problems that account for the specific features of dynamic controllers.

2. STATEMENT OF THE PROBLEM

We will consider a controllable object of the nth order

$$\dot{x} = Ax + bu, \quad x(0) = x_0, \tag{1}$$

the input of which receives on the time interval $T = [0, t^*]$ the control action

$$u = d'y, \tag{2}$$

produced by a dynamic controller of the rth order

$$\dot{y} = By + fv, \quad y(0) = y_0. \tag{3}$$

Here, $x = x(t)$ is the n-vector of the state of the controllable object at the instant of time t, $y = y(t)$ is the r-vector of the state of the controller, x_0 and y_0 are the initial states of the controllable object and the controller, respectively, $u = u(t)$ is a value of the scalar control action, $v = v(t)$ is a value of the scalar control signal, and $A \in R^{n\times n}$, $b \in R^n$, $d \in R^r$, $B \in R^{r\times r}$, $f \in R^r$.

We will assume that the control signal $v(t)$, $t \in T$, on the time intervals in which the control action does not take boundary values is a discrete function with the quantization period $h = t^*/N$ (N is a natural number): $v(t) = v(kh)$, $t \in [kh, (k+1)h[\,$, $k = \overline{0, N-1}$. The class of control signals on the remaining time intervals will be defined later on.

Let $Y \subset R^r$ be a domain within which transient processes in the controller occur. We assume that for all $y \in Y$ satisfying the conditions

$$d'y = L, \quad d'By = 0, \ldots, d'B^{k-1}y = 0, \tag{4}$$

the following inequality is met:

$$|d'B^k y / d'B^{k-1} f| < M. \tag{5}$$

It is easy to verify that relations (4) and (5) for the controller

$$u^{(r)} + \alpha_r u^{(r-1)} + \ldots + \alpha_2 \dot{u} + \alpha_1 u = v \tag{6}$$

will have the form

$$u = L, \quad \dot{u} = 0, \ldots, u^{(r-1)} = 0, \quad |\alpha_1 L| < M.$$

To each control signal $v(\cdot) = (v(t), t \in T)$ there corresponds a single trajectory $y(\cdot) = (y(t), t \in T)$ of the controller (3), a control action $u(\cdot) = (u(t),\ t \in T)$, and a trajectory $x(\cdot) = (x(t),\ t \in T)$ of the controllable object (1).

The functions $v(\cdot)$, $y(\cdot)$, $u(\cdot)$, and $x(\cdot)$ will be called admissible if

$$\begin{gathered} |v(t)| \le M, \quad |u(t)| \le L, \quad t \in T; \quad Hx(t^*) = g; \\ (H \in R^{m\times n}, \quad \operatorname{rank} H = m < n, \quad g \in R^m). \end{gathered} \tag{7}$$

We will assume that the constraint $|u(t)| \le L,\ t \in T$, of the controller is of depth k (order), i.e., there exist relations

$$d'f = 0, \quad d'Bf = 0, \ldots, d'B^{k-2}f = 0, \quad d'B^{k-1}f \ne 0.$$

In this case, at the instant $t \in T$, the derivatives of the control action $u(t)$, $\dot{u}(t) = d'\dot{y}(t) = d'By(t), \ldots, u^{(k-1)}(t) = d'B^{k-1}y(t)$ do not depend on the control signal $v(t)$, but the derivative $u^{(k)}(t) = d'B^k y(t) + d'B^{k-1} f v(t)$ depends on that signal.

The phase constraint $|u(t)| \le L,\ t \in T$, of the controller (6) is of depth r (order).

The quality of an admissible control signal $v(\cdot)$ (a program) will be defined by the functional

$$J(v) = c'x(t^*). \tag{8}$$

An optimal signal $v^0(\cdot)$ (an optimal program) will be specified by the equality

$$J(v^0) = \max J(v), \tag{9}$$

where the maximum is taken over all programs.

In developing optimal programs, we will restrict the discussion only to such control actions as go out to the boundaries of an imposed constraint a finite number of times. In this case, a control action either can come in contact with the boundaries at certain instants of time or can “bed” on the boundaries in some time intervals. We will differentiate two types of going out of a control action to sections of bedding: the soft and the rigid going out. At the soft going out to a boundary (soft saturation of a control action), a phase trajectory of the controller remains continuous at the instant of the going out. The decision as to the soft going out is taken in addition to phase constraints once the control action remains in a prescribed small neighborhood of the boundary in the interval between the last two contacts with the boundary.[2] The soft going out is made possible on account of the choice of an appropriate control signal. The class of control actions in the sections of saturation follows from the conditions of holding a control action at the boundary of its possible values. At the rigid going out, a phase trajectory of the controller at the instant of the going out to the boundary exhibits a jump. The retention of the control action at the boundary is ensured by additional means (external ones with respect to problems (1)–(3), (7)–(9)). In this case, a control signal is only used to afford optimal transient processes between instants of the departure from a boundary and the going out to the boundary of a phase trajectory of the controller. Thus, at the soft going out, control signals are chosen so as to avoid the violation of the constraints on a control action and at the rigid going out, the care for that procedure rests on other (external) means. A similar pattern also takes place in the departure from the boundaries of phase constraints.

The described class of admissible controls enables us to obviate the Fuller modes [3], which are of interest to mathematicians but hardly hold any practical significance.

Optimal programs in this work play an auxiliary role. The main aim of the work is to develop the method of implementing optimal feedback-type control signals in the real time mode. These signals are defined by means of optimal programs.

Let us insert the problem (1)–(3), (7)–(9) into a set of the problems

$$\begin{gathered} c'x(t^*) \to \max; \quad \dot{x} = Ax + bu, \quad x(\tau) = z; \\ u = d'y, \quad \dot{y} = By + fv, \quad y(\tau) = p; \\ Hx(t^*) = g; \quad |v(t)| \le M, \quad |u(t)| \le L, \quad t \in T^\tau = [\tau, t^*], \end{gathered} \tag{10}$$

which depends on the scalar $\tau \in T_h = \{0, h, \ldots, t^* - h\}$, the n-vector z, and the r-vector p. Let $v^0(t \mid \tau, z, p),\ t \in T^\tau$, be an optimal program of the problem (10) for the position (τ, z, p)

[2] Strictly speaking, in this case suboptimal controls result that approximate the optimal control with any specified accuracy. But we will not make changes in the introduced definition of the optimal control, assuming that the solutions obtained in the work are “practically” optimal.

and $Q_\tau \subset R^{n+r}$ be a set of the states (z, p) of the control system (1)–(3). For these states, the problem (10) has a solution. We will call the function

$$v^0(\tau, z, p) = v^0(\tau \mid \tau, z, p), \quad (z, p) \in Q_\tau, \quad \tau \in T_h, \tag{11}$$

an optimal control signal of the feedback type for (1)–(3), (7)–(9).

The trajectory of a nonlinear control system that is closed by optimal feedback (11) and is found to be under the action of the disturbance $w(t),\ t \in T$,

$$\dot{x} = Ax + bu + w, \quad x(0) = x_0; \quad u = d'y; \quad \dot{y} = By + fv^0(t, x, y), \quad y(0) = y_0, \tag{12}$$

will be taken to mean the trajectory of the linear equations

$$\dot{x} = Ax + bu + w, \quad x(0) = x_0; \quad u = d'y; \quad \dot{y} = By + fv(t), \quad y(0) = y_0,$$

with $v(t) = v(jh, x(jh), y(jh)),\ t \in [jh, (j+1)h[,\ j = \overline{0, N-1}$.

Deriving the function (11) in the explicit form (synthesis of an optimal system in the classical sense) involves a very complex problem. In this work, we solve the problem of synthesis of optimal systems on the basis of the statement suggested in [6], i.e., we develop an algorithm for implementing optimal feedback (11) in the real time mode. In executing such an approach, methods of calculating optimal programs play a decisive role.

3. CALCULATION OF AN OPTIMAL PROGRAM OF THE AUXILIARY PROBLEM

As the basis for the method of calculating an optimal program $v^0(t),\ t \in$, of the problem (1)–(3), (7)–(9), we will set up the following three procedures: (1) the calculation of an optimal program $\widetilde{v}^0(t),\ t \in T$, for the auxiliary OC problem with intermediate phase constraints; (2) the correlation of an optimal program of the auxiliary problem; and (3) the check of the developed program for optimality.

In this section, we will describe the first procedure. From the set T_h, we will select a finite number of isolated instants $S = \{s_1, s_2, \ldots, s_l\}$ and introduce an auxiliary OC problem with the intermediate phase constraints

$$\begin{gathered} c'x(t^*) \to \max; \quad \dot{x} = Ax + bu, \quad x(0) = x_0; \\ u = d'y; \quad \dot{y} = By + fv, \quad y(0) = y_0; \\ Hx(t^*) = g; \quad |v(t)| \le M, \quad t \in T; \quad |u(s)| \le L, \quad s \in S. \end{gathered} \tag{13}$$

The fast algorithms for the solution of problems of the form (13) are set out in [5]. We will present the necessary elements of the dual method [7] in the form that is convenient for the solution of the problem under consideration. First, we will write the problem in the equivalent functional form

$$\begin{aligned} &\sum_{t \in T_h} c(t)v(t) \to \max; \quad \sum_{t \in T_h} h_i(t)v(t) = \widetilde{g}_i, \quad i = \overline{1, m}; \\ &L_*(s) \le \sum_{t \in T_h} q(s, t)v(t) \le L^*(s), \quad s \in S; \qquad |v(t)| \le M, \quad t \in T_h. \end{aligned} \tag{14}$$

Here,

$$c(t) = \int\limits_t^{t+h} \varphi_c'(\vartheta) f d\vartheta; \quad h_i(t) = \int\limits_t^{t+h} \varphi_{hi}'(\vartheta) f d\vartheta, \quad i = \overline{1, m},$$

where h_i is the ith row of the matrix H, and

$$q(s,t) = \int\limits_{t}^{t+h} \varphi'(s,\vartheta) f d\vartheta, \quad t < s; \quad q(s,t) = 0, \quad t \geq s; \tag{15}$$

$$\widetilde{g}_i = g_i - \psi'_{hi}(0)x_0 - \varphi'_{hi}(0)y_0; \quad L_*(s) = -L - \varphi'(s,0)y_0,$$
$$L^*(s) = L - \varphi'(s,0)y_0;$$

where $(\psi_c(t),\ \varphi_c(t)),\ t \in T$, and $(\psi_{hi}(t),\ \varphi_{hi}(t)),\ t \in T$, are solutions of the adjoint equations

$$\dot{\psi}' = -\psi' A, \quad \dot{\varphi}' = -\varphi' B - \psi' b d' \tag{16}$$

under the initial conditions, respectively,

$$\psi(t^*) = c, \quad \varphi(t^*) = 0; \quad \psi(t^*) = h_i, \quad \varphi(t^*) = 0;$$

where $\varphi(s,t),\ t \leq s,\ s \in S,\ t \in T$ is the solution of the adjoint equation of the controller

$$\dot{\varphi}' = -\varphi' B, \quad \varphi(s,s) = d, \quad s \in S. \tag{17}$$

The problem (14) is the interval problem of linear programming (LP). The standard LP methods for its solution are low-effective in view of specific features of the matrix of conditions. The subsequent structures aim at performing the dynamic implementation of the dual adaptive method [7] that takes into account to a maximum extent the character of (15) in regard to elements of the problem (14).

The basic element of the adaptive method is a support. Following [7], we will separate the subset S_{sup} from the set S. We will isolate the subset T_{sup} from the set T_h in such a way that $|T_{\text{sup}}| = |S_{\text{sup}}| + m$. Let us compose the $|T_{\text{sup}}| \times |T_{\text{sup}}|$-matrix

$$D_{\text{sup}} = \begin{pmatrix} \begin{matrix} h_i(t), \\ i = \overline{1,m}; \end{matrix} & t \in T_{\text{sup}}, \\ & \\ \begin{matrix} q(s,t), \\ s \in S_{\text{sup}} \end{matrix} & t \in T_{\text{sup}}, \end{pmatrix}.$$

The aggregate $K_{\text{sup}} = \{S_{\text{sup}}, T_{\text{sup}}\}$ with $S_{\text{sup}} \neq \emptyset$ and $T_{\text{sup}} \neq \emptyset$ will be called a support if $\det D_{\text{sup}} \neq 0$. If $K_{\text{sup}} = \{S_{\text{sup}} = \emptyset,\ T_{\text{sup}} = \emptyset\}$, the aggregate K_{sup} is an (empty) support by definition. We introduce the following designations: $S_{\text{un}} = S \setminus S_{\text{sup}}$ and $T_{\text{un}} = T_h \setminus T_{\text{sup}}$, where the subscript un implies "unsupported."

The support K_{sup} is accompanied by the following statements:

(1) The function of potentials $\nu(s),\ s \in S$, and the m-vector of potentials $\mu = (\mu_1, \ldots, \mu_m)$. We will develop them starting from $\nu_{\text{un}} = \nu(S_{\text{un}}) = (\nu(s),\ s \in S_{\text{un}}) = 0$. The vector μ and support components $\nu_{\text{sup}} = (\nu(s),\ s \in S_{\text{sup}})$ will be estimated as a solution of the equation

$$(\mu, \nu_{\text{sup}})' D_{\text{sup}} = c'_{\text{sup}},$$

where $c_{\text{sup}} = c(T_{\text{sup}}) = (c(t),\ t \in T_{\text{sup}})$. In the case of an empty support, we assume that $\nu(s) = 0$, $s \in S$, and $\mu = 0$.

(2) The cotrajectory $(\psi(t),\ \varphi(t)),\ t \in T$, namely, the solution of the adjoint Eqs. (16) under the initial conditions $\psi(t^*) = c - H'\mu,\ \varphi(t^*) = 0$, which have jumps with respect to the component $\varphi(t)$, $t \in T$, at intermediate instants of time $s \in S$:

$$\varphi(s-0) = \varphi(s+0) - d\nu(s), \quad s \in S.$$

(3) The cocontrol $\delta_h(t),\ t \in T_h$,

$$\delta_h(t) = \int\limits_t^{t+h} \delta(\tau)d\tau, \quad t \in T_{\text{un}}, \quad (\delta(\tau) = \varphi'(\tau)f); \quad \delta_h(t) = 0, \quad t \in T_{\text{sup}}.$$

For an empty support, we have $\delta_h(t) = c(t) = \int\limits_t^{t+h} \varphi_c(\tau)d\tau,\ t \in T_{\text{un}} = T_h$.

(4) The pseudosignal $\omega(t),\ t \in T$, and the pseudoaction $\zeta(t),\ t \in T$. We will begin to develop them by prescribing unsupported values of the pseudosignal $\omega(t),\ t \in T_{\text{un}}$,

$$\omega(t) = M \operatorname{sgn} \delta_h(t), \quad \text{if} \quad \delta_h(t) \neq 0;$$
$$\omega(t) \in [-M, M], \quad \text{if} \quad \delta_h(t) = 0, \quad t \in T_{\text{un}},$$

and support values of the pseudoaction $\zeta(t), \quad t \in S_{\text{sup}}$,

$$\zeta(t) = L \operatorname{sgn} \nu(t), \quad \text{if} \quad \nu(t) \neq 0;$$
$$\zeta(t) \in [-L; L], \quad \text{if} \quad \nu(t) = 0, \quad t \in S_{\text{sup}}.$$

In the case of an empty support, we put $\zeta(t) = 0,\ t \in T_h$. The support components $\omega_{\text{sup}} = (\omega(t),\ t \in T_{\text{sup}})$ of a pseudosignal will be found from a system $|S_{\text{sup}}| + m$ of the equations

$$D_{\text{sup}}\omega_{\text{sup}} = \begin{pmatrix} g_i - h_i' x_0(t^*), \\ i = \overline{1,m}; \\ \zeta(s) - d' y_0(s), \\ s \in S_{\text{sup}} \end{pmatrix},$$

where $x_0(t)$ and $y_0(t),\ t \in T$, are trajectories of the control system (1)–(3) and the controller (3) with $v(t) = \omega(t)$, $t \in T_{\text{un}}$, and $v(t) = 0$, $t \in T_{\text{sup}}$.

If the support K_{sup} is empty, then we do not build $\omega(t),\ t \in T_{\text{sup}}$. We will estimate unsupported components of the pseudoaction $\zeta(t),\ t \in T^*_{\text{un}} = T_h \setminus S_{\text{sup}}$, from the formula

$$\zeta(t) = d' y_*(t), \quad t \in T^*_{\text{un}},$$

where $y_*(t),\ t \in T$ is a solution of Eq. (3) with $v(t) = \omega(t)$, $t \in T$.

The support K_{sup} will be called regular if $\nu(s) \neq 0,\ s \in S_{\text{sup}}$, and $\delta_h(t) \neq 0,\ t \in T_{\text{un}}$. The regular support is followed by a unique pseudosignal and a unique pseudoaction.

Considering [7], we will use a support primarily for the identification of optimal and ε-optimal programs.

Theorem 1 (maximum principle). *To afford the optimality of the program $v(\cdot)$, it is necessary and sufficient to have a support $K_{\text{sup}} = \{S_{\text{sup}}, T_{\text{sup}}\}$ such that the following conditions are met on the function of potentials $\nu(s),\ s \in S$, and the cocontrol $\delta_h(t),\ t \in T_h$, which accompany this support:*

(1) *the condition of a maximum in the action*

$$\nu(s)u(s) = \max_{-L \leq u \leq L} \nu(s)u, \quad s \in S_{\text{sup}};$$

(2) *the condition of a maximum in the signal*

$$\delta_h(t)v(t) = \max_{-M \leq v \leq M} \delta_h(t)v, \quad t \in T_{\text{un}}.$$

The support K_{sup} that identifies an optimal program will be called optimal; optimal elements accompany this support.

Theorem 2 (principle of ε-maximum). *To ensure the ε-optimality of the program $v(\cdot)$ at any $\varepsilon \geq 0$, it is necessary and sufficient to have a support $K_{\text{sup}} = \{S_{\text{sup}}, T_{\text{sup}}\}$ such that the following conditions are met on the function of potentials $\nu(s)$, $s \in S$, and the cocontrol $\delta_h(t)$, $t \in T_h$, which accompany this support:*

(1) *the condition of a quasimaximum in the action*

$$\nu(s)u(s) = \max_{-L \leq u \leq L} \nu(s)u - \varepsilon_u(s), \quad s \in S_{\text{sup}};$$

(2) *the condition of a quasimaximum in the signal*

$$\delta_h(t)v(t) = \max_{-M \leq v \leq M} \delta_h(t)v - \varepsilon_v(t), \quad t \in T_{\text{un}};$$

(3) *the condition of the ε-accuracy*

$$\sum_{s \in S_{\text{sup}}} \varepsilon_u(s) + \sum_{t \in T_{\text{un}}} \varepsilon_v(t) \leq \varepsilon.$$

Theorem 3 (criterion of support optimality). *To ensure the optimality of the support K_{sup}, it is necessary and sufficient to fulfill some pseudoaction and some pseudosignal, which accompany this support:*

$$|\zeta(s)| \leq L, \quad s \in T^*_{\text{un}}; \quad |\omega(t)| \leq M, \quad t \in T_{\text{sup}}.$$

The pseudosignal $\omega(t)$, $t \in T$, and the pseudoaction $\zeta(t)$, $t \in T$, that accompany an optimal support are the optimal control signal and the optimal control action: $\widetilde{v}^0(t) = w(t)$ and $\widetilde{u}^0(t) = \zeta(t)$, $t \in T$.

We will describe the *dual method* of calculating an optimal program control of the problem (13). The method involves a sequential transformation of supports with the aim to produce an optimal support. Let a certain (possibly empty) support K_{sup} of the problem (13) be known. Relying on the support K_{sup}, we will develop a pseudosignal $\omega(t)$, $t \in T$, and a pseudoaction $\zeta(t)$, $t \in T$, which attend the support.

Let us introduce the designations: $S^-_{\text{un}} = \{t \in S_{\text{un}} : \zeta(t) \notin [-L, L]\}$, $T^-_{\text{sup}} = \{t \in T_{\text{sup}} : \omega(t) \notin [-M, M]\}$, and $K^- = S^-_{\text{un}} \cup T^-_{\text{sup}}$. If $K^- = \emptyset$, then K_{sup} is an optimal support, $\widetilde{v}^0(t) = \omega(t)$, $t \in T$ is an optimal control signal, and $\widetilde{u}^0(t) = \zeta(t)$, $t \in T$ is an optimal control action.

If $K^- \neq \emptyset$, we select an arbitrary element from K^-. Two cases are possible: (1) the element belongs to the set S^-_{un}: $s_0 \in S^-_{\text{un}}$ and (2) the element is from the set T^-_{sup}: $t_0 \in T^-_{\text{sup}}$.

(I) ($s_0 \in S^-_{\text{un}}$). We will start from the development of an increment $\Delta\nu(t)$, $t \in S_{\text{un}}$, of an unsupported component of the function of potentials:

$$\Delta\nu(s_0) = \begin{cases} 1, & \text{if } \zeta(s_0) > L \\ -1, & \text{if } \zeta(s_0) < -L, \end{cases} \qquad \Delta\nu(t) = 0, \quad t \in S_{\text{un}} \setminus s_0. \tag{18}$$

Next, we will find increments $\Delta\mu$ and $\Delta\nu_{\text{sup}}$ of the vector of potentials and support components of the function of potentials, respectively, from the equation

$$(\Delta\mu, \Delta\nu_{\text{sup}})' D_{\text{sup}} = (\Delta\delta_h^{s_0}(t), \quad t \in T_{\text{sup}})', \tag{19}$$

where $\Delta\delta_h^{s_0}(t) = \int\limits_t^{t+h} \Delta\delta^{s_0}(\tau)d\tau$, $t \in T_{\text{un}}$; $\Delta\delta^{s_0}(\tau) = f'\Delta\varphi^{s_0}(\tau)$; $\Delta\varphi^{s_0}(t)$, $t \in T$, is a component of the solution of adjoint Eqs. (16) with initial conditions $\psi(t^*) = 0$ and $\varphi(t^*) = 0$, which has a jump at the instant of time $s_0 \in S$:

$$\varphi(s_0 - 0) = \varphi(s_0 + 0) - d\Delta\nu(s_0).$$

Increments $\Delta\delta_h(t),\ t \in T_h$, of the cocontrol will be found from the formulas

$$\Delta\delta_h(t) = \int_t^{t+h} \Delta\delta(\tau)d\tau, \quad t \in T_{\text{un}}, \quad (\Delta\delta(\tau) = \Delta\varphi'(\tau)f), \quad \Delta\delta_h(t) = 0, \quad t \in T_{\text{sup}}, \tag{20}$$

where $\Delta\varphi(t),\ t \in T$, is a component of the solution of Eqs. (16) with the initial conditions

$$\psi(t^*) = -H'\Delta\mu, \quad \varphi(t^*) = 0, \tag{21}$$

which has jumps at intermediate instants of time $s \in S$:

$$\varphi(s-0) = \varphi(s+0) - d\Delta\nu(s), \quad s \in S. \tag{22}$$

In the motion of

$$\begin{gathered} \mu(\sigma) = \mu + \sigma\Delta\mu; \quad \nu(t|\sigma) = \nu(t) + \sigma\Delta\nu(t), \quad t \in S; \\ \delta_h(t|\sigma) = \delta_h(t) + \sigma\Delta\delta_h(t), \quad t \in T_h; \quad \sigma \geq 0; \end{gathered} \tag{23}$$

from $(\mu, \nu(\cdot), \delta_h(\cdot))$ along one of the directions (18)–(20), the initial rate α^1 of a decrease in the dual objective function is equal to $\alpha^1 = L - \zeta(s_0)$ if $\zeta(s_0) > L$ or $\alpha^1 = \zeta(s_0) + L$ if $\zeta(s_0) < -L$.

With an increase in $\sigma \geq 0$, new zeros of functions (23) will appear:

$$\begin{aligned} \sigma_u(s) &= \begin{cases} -\nu(s)/\Delta\nu(s), & \text{if } \ \nu(s)\Delta\nu(s) < 0 \\ \infty, & \text{if } \ \nu(s)\Delta\nu(s) \geq 0, \end{cases} \qquad s \in S_{\text{sup}}; \\ \sigma_v(t) &= \begin{cases} -\delta_h(t)/\Delta\delta_h(t), & \text{if } \ \delta_h(t)\Delta\delta_h(t) < 0 \\ \infty, & \text{if } \ \delta_h(t)\Delta\delta_h(t) \geq 0, \end{cases} \qquad t \in T_{\text{un}}. \end{aligned} \tag{24}$$

At $K_{\text{sup}} = \emptyset$, we set $\sigma_u(s) = \infty,\ s \in S_{\text{sup}}$.

We will arrange in the ascending sequence the finite numbers from (24): $\sigma^1 < \sigma^2 < \ldots < \sigma^{k_*}$ (the absence of these numbers points to the inconsistency of the constraints of the problem (13)).

In the motion (23), the dual objective function changes linearly with respect to σ between adjacent zeros of the cocontrol $\sigma^k < \sigma^{k+1}$ from (24). Its rate α^k undergoes a jump in this section at the point σ^{k+1}

$$\begin{gathered} \Delta\alpha^k = \sum_{s \in S^k} 2L\ |\Delta\nu(s)| + \sum_{t \in T^k} 2Mh\ |\Delta\delta_h(t)|, \\ S^k = \{s \in S_{\text{sup}} : \nu(s|\sigma^k) = 0\}, \quad T^k = \{t \in T_{\text{un}} : \Delta_h(t|\sigma^k) = 0\}. \end{gathered}$$

Thus, $\alpha^{k+1} = \alpha^k + \Delta\alpha^k$.

Let us denote by k_0 a superscript such that $\alpha^{k_0} < 0$ and $\alpha^{k_0+1} \geq 0$. The number $\sigma^* = \sigma^{k_0}$ is said to be a short dual step if $k_0 = 1$, otherwise σ^* is a long dual step. We will select an arbitrary element from the set $S^{k_0} \cup T^{k_0}$. Two cases are possible:

(Ia) the element belongs to the set S^{k_0}: $s_* \in S^{k_0}$,

(Ib) the element belongs to the set T^{k_0}: $t_* \in T^{k_0}$.

Let us form components of the new support $\overline{K}_{\text{sup}} = \{\overline{S}_{\text{sup}}, \overline{T}_{\text{sup}}\}$: (Ia) $\overline{S}_{\text{sup}} = (S_{\text{sup}} \setminus s_*) \cup s_0$, $\overline{T}_{\text{sup}} = T_{\text{sup}}$ and (Ib) $\overline{S}_{\text{sup}} = S_{\text{sup}} \cup s_0$, $\overline{T}_{\text{sup}} = T_{\text{sup}} \cup t_*$.

(II) ($t_0 \in T^-_{\text{sup}}$). We will start with the formation of the increment $\Delta\delta_h(t),\ t \in T_{\text{sup}}$,

$$\Delta\delta_h(t_0) = \begin{cases} 1, & \text{if } \ \omega(t_0) > M \\ -1, & \text{if } \ \omega(t_0) < -M, \end{cases} \qquad \Delta\delta_h(t) = 0, \quad t \in T_{\text{sup}} \setminus t_0. \tag{25}$$

To that increment there correspond increments of unsupported components of the function of potentials:

$$\Delta\nu(s) = 0, \quad s \in S_{\text{un}}. \tag{26}$$

The increments $\Delta\mu$ and $\Delta\nu_{\text{sup}}$ of the vector of potentials and the components of the function of potentials, respectively, will be found from the equation

$$(\Delta\mu, \Delta\nu_{\text{sup}})' D_{\text{sup}} = -(\Delta\delta_h(t), \quad t \in T_{\text{sup}})'. \tag{27}$$

Finally, increments $\Delta\delta_h(t),\ t \in T_{\text{un}}$, will be found from the formula

$$\Delta\delta_h(t) = \int\limits_t^{t+h} \Delta\delta(\tau)d\tau, \quad t \in T_{\text{un}}, \quad (\Delta\delta(\tau) = \Delta\varphi'(\tau)f), \tag{28}$$

where $\Delta\varphi(t),\ t \in T$, is a component of the solution of the adjoint Eqs. (16) with the initial conditions (21), which has the jumps (22) in intermediate instants of time $s \in S$.

In the motion (23) along (25)–(28), the initial rate α^1 of the dual objective function is equal to $\alpha^1 = M - \omega(t_0)$ when $\omega(t_0) > M$ or $\alpha^1 = \omega(t_0) + M$ when $\omega(t_0) < -M$.

Next, we act in the same way as in Case (I).

The new support has the form $\overline{K}_{\text{sup}} = \{\overline{S}_{\text{sup}}, \overline{T}_{\text{sup}}\}$, where (Ia) $\overline{S}_{\text{sup}} = S_{\text{sup}} \setminus s_*$, $\overline{T}_{\text{sup}} = T_{\text{sup}}$ and (Ib) $\overline{S}_{\text{sup}} = S_{\text{sup}}$, $\overline{T}_{\text{sup}} = (T_{\text{sup}} \setminus t_0) \cup t_*$.

In each case, the support $\overline{K}_{\text{sup}}$ is followed by (1) the vector of potentials $\overline{\mu} = \mu + \sigma^*\Delta\mu$; (2) the function of potentials $\overline{\nu}(s) = \nu(s) + \sigma^*\Delta\nu(s),\ s \in S$; and (3) the cocontrol $\overline{\delta}_h(t) = \delta_h(t) + \sigma^*\Delta\delta_h(t)$, $t \in T_h$. The fast implementation of the operations cited above is described in [5].

According to [7], the process involving the suggested dual method comes to an end if in the course of iterations, only regular supports occur. The modification of this method is also described in [7], which is finite for any problem (13). The method completes the operation on forming an optimal support and a pseudosignal and a pseudoaction that accompany the support, which will be the optimal control signal and the optimal control action of the auxiliary problem.

4. DEVELOPMENT OF AN OPTIMAL PROGRAM OF THE INITIAL PROBLEM

In the general case, the optimal program $\widetilde{v}^0(t),\ t \in T$, of the auxiliary problem (13) will not satisfy the phase constraints of the initial problem (1)–(3), (7)–(9). We will evolve an optimal program of the initial problem on the basis of the procedure of correcting the optimal program of the auxiliary problem. This procedure consists in the sequential change of constraints of the auxiliary problem. We will previously introduce the necessary definitions.

Definition 1. The time interval $T^* = [\tau_*, \tau^*]$ will be called a section of saturation (of a control action) if[3]

(1) $u(t) = L, \quad t \in T^*$,

(2) on account of the solution $\widetilde{v}^0(t)$, $t \in T$, of the extremal problem (13) with the constraint $u(t) = L$, $t \in T^*$, the inequalities $\widetilde{v}^0(\tau_* - h) > 0$ and $\widetilde{v}^0(\tau^*) < 0$ are fulfilled,

(3) on account of the solution $\widetilde{v}^0(t),\ t \in T$, of the extremal problem (13) with the constraint $u(t) = L$, $t \in [\tau_* + h, \tau^* - h]$, the inequalities $\widetilde{v}^0(\tau_*) < 0$ and $\widetilde{v}^0(\tau^* - h) > 0$ are fulfilled.

[3] For definiteness, we further consider only the upper bound of possible values of the control action.

Definition 2. The procedure of regularization of the bedding section $u(t) = L,\ t \in T^*$ will be called the construction from the last saturation section or its removal. This procedure consists in the following. The bedding section is extended by an elementary interval if the appropriate inequality from Condition (2) in Definition 1 is not fulfilled or contracted if Condition (3) in Definition 1 is not fulfilled.

Definition 3. The instant $s_* \in T^*$ will be called a point of (approximate) contact (accurate to h) by a control action of the boundary of possible values if

$$u(s_* - h) < L, \quad u(s_*) \geq L, \quad u(s_* + h) < L.$$

Definition 4. The retaining control signal on the interval $[s_*, s_* + h[$ will be called a function $v^*(t) = -d'B^k y^*(t)/d'B^{k-1}f,\ t \in [s_*, s_* + h[$, where $y^*(t),\ t \in [s_*, s_* + h[$, is a solution of the differential equation $\dot{y} = (B - fd'B^k)y/d'B^{k-1}f$ with initial conditions $d'y(s_*) = L$ and $d'By(s_*) = 0, \ldots, d'B^{k-1}y(s_*) = 0$.

For the controller (6), we obtain $v^*(t) = \alpha_r u_*^{(r-1)}(t) + \ldots + \alpha_1 u_*(t)$ and $u_*(t) \equiv L,\ t \in [s_*, s_* + h[$, which is a solution of the differential equation $u^{(r)}(t) = 0$ with initial conditions $u(s_*) = L$ and $\dot{u}(s_*) = 0, \ldots, u^{(r-1)}(s_*) = 0$.

We will begin to carry out the correction procedure from the calculation of an optimal program $\widetilde{v}^0(t), t \in T$, of the auxiliary problem (13) in the case of the selected set $S \subset T_h$ of intermediate instants. Without the loss of generality, we can assume that along this program, the phase constraint is active at all points $s \in S$ $(\widetilde{u}^0(s) = L,\ s \in S)$ and there are no bedding sections $\widetilde{u}^0(t) \equiv L$, $t \in T^*$. We denote by $s_* \in T_h$ the least instant at which the phase constraint $\widetilde{u}^0(s_*) > L$ breaks down.

Let us select[4] a fairly small number $\varepsilon > 0$ and check at the point s_* the condition of the transition to the procedure of laying (of a control action on the boundary of possible values):

$$d'y(s_*) - L \leq \varepsilon; \quad |d'B^i y(s_*)| \leq \varepsilon, \quad i = \overline{1, k-1}; \qquad (29)$$
$$|d'B^k y(s_*)/d'B^{k-1}f| < M.$$

For the controller (6), conditions (29) have the form

$$u(s_*) - L \leq \varepsilon; \quad |u^{(i)}(s_*)| \leq \varepsilon, \quad i = \overline{1, r-1};$$
$$|\alpha_r u^{(r-1)}(s_*) + \ldots + \alpha_1 u(s_*)| < M.$$

If conditions (29) are met, we will solve the auxiliary problem (13) with additional constraints at the point s_*:

$$d'y(s_*) = L; \quad d'B^i y(s_*) = 0, \quad i = \overline{1, k-1}; \quad |d'B^k y(s_*)/d'B^{k-1}f| \leq M. \qquad (30)$$

These relations for the controller (6) assume the form

$$u(s_*) = L; \quad u^{(i)}(s_*) = 0, \quad i = \overline{1, r-1}; \quad |\alpha_r u^{(r-1)}(s_*) + \ldots + \alpha_1 u(s_*)| \leq M.$$

On the basis of the solution $\widetilde{v}^0(t),\ t \in T$, of the new problem, we will verify the following conditions: (a) $-d'B^k y(s_*)/d'B^{k-1}f = M$ and (b) the smallness of a disturbance, i.e., the points of contact on $[0, s_*]$ did not change.

[4] The rule of the choice of the number ε is given below.

With the fulfillment of these conditions, we will pass on to the procedure of laying (see below). If at least one of these conditions breaks down, we will reduce ε and begin to perform the correction procedure again.

Let us examine the situation in which the conditions of the transition to the laying procedure (29) are not met. We will check the inequality

$$\widetilde{u}^0(s_* + h) \leq L. \tag{31}$$

With its fulfillment, we consider that s_* is the point of contact. We will find a new point s_* of the violation of a phase constraint.

If inequality (31) is not valid, then we will extend the set $S := S \cup \{s_*, s_* + h\}$ and solve the problem (13) with the new set S. Let s_* identify the least instant from the set $T_h \cap [0, s_*]$ in which the phase constraint is broken and let us check in it the conditions of the transition to the laying procedure (29).

We assume that as a result of the performed corrections, a control $\widetilde{u}^0(t),\ t \in T$, is set up and an instant $s_* \in T_h$ is found, such that (1) $\widetilde{u}^0(s_*) > L$ and (2) in the segment $[0, s_*]$, a phase constraint is fulfilled, excepting, possibly, the fulfillment of contact points; in none of the contact points, the conditions of the transition to the laying procedure are met; and the control$\widetilde{u}^0(t),\ t \in T$, has saturation sections (to avoid cumbersome calculations, we assume that only one such section exists).

We will check at the point s_* the conditions of the transition to the laying procedure (29). In the case of their fulfillment, we will solve the auxiliary problem (13) by supplementing the available constraints (30) with the additional constraints

$$\begin{gathered} |d'B^k y(\tau_* - h)/d'B^{k-1} f| \leq M; \quad d'y(\tau_*) = L; \quad d'B^i y(\tau_*) = 0, \quad i = \overline{1, k-1}; \\ d'y(t) \equiv L, \quad t \in T^*; \quad |d'B^k y(\tau^*)/d'B^{k-1} f| \leq M. \end{gathered} \tag{32}$$

Relations (32) can be readily written for the controller (6).

On the basis of the solution $\widetilde{v}^0(t),\ t \in T$, of the new problem, we will verify the following conditions: (a) $-d'B^k y(s_*)/d'B^{k-1} f = M$ and (b) the smallness of a disturbance, in which case the contact points are retained and the saturation sections do not change on $[0, s_*]$.

With the disruption at least one of these conditions, we decrease ε and perform the correction procedure again. Otherwise, we pass on to the laying procedure (see below).

We will consider the situation in which the conditions of the transition to the laying procedure (29) are not met. Let us verify inequality (31). In the case of its fulfillment, we assume that s_* is a contact point. We will find a new point s_* of the violation of the phase constraint.

If inequality (31) is not met, we will extend the set $S := S \cup \{s_*, s_* + h\}$ and solve the problem (13) with the new set S and with the additional constraints (32).

Let us check the fulfillment of the phase constraint on the interval $[0, s_*]$. In the case of its violation, we denote by s_* the least instant from the set $T_h \cap [0, s_*]$ in which the phase constraint is broken. If to the left of s_* there is at least one laying section, we will perform for it the regularization procedure and find again the least instant from $T_h \cap [0, s_*]$ of the breaking of the phase constraint. Let us check at the point s_* the conditions of the transition to the laying procedure (29).

Procedure of Laying of a Control Action on the Boundaries of Possible Values

We will lay a control action on the interval $[s_*, s_* + h[$, using a retaining control signal $v^*(t)$, $t \in [s_*, s_* + h[$, and solve the auxiliary problem (13) with additional constraints: $d'y(s_*) = L$; $d'B^i y(s_*) = 0$, $i = \overline{1, k-1}$; $v(t) = v^*(t)$, $t \in [s_*, s_* + h[$; and $|d'B^k y(s_* + h)/d'B^{k-1} f| \leq M$.

Let us check the fulfillment of a phase constraint on the interval $[0, s_*]$. In the case of its violation, we denote by s_* the least instant from the set $T_h \cap [0, s_*]$ in which the phase constraint is disturbed. If to the left of s_* there is any of the laying sections T^* or $[s_*, s_* + h[$, then we will perform for it the regularization procedure and find again the least instant from $T_h \cap [0, s_*]$ of the violation of the phase constraint. Let us verify at the point s_* the condition of the transition to the laying procedure (29).

Note 1. The described procedure of correction leads to the development of a control action in a finite number of steps, which everywhere, apart from a finite number of the points of approximate contact, will satisfy the constraint $|u(t)| \leq L$, $t \in T$. To increase the accuracy of the fulfillment of the phase constraint, it is sufficient to lay a finer-mesh net in the neighborhood of the indicated points. Another way involves the use, by analogy with [5], of the procedure of sizing, which affords the accuracy of contact by the control action of the boundaries of possible values in the neighborhood of the indicated points of approximate contact.

Note 2. The number $\varepsilon > 0$ that ensures the transition to the procedure of laying depends on h and may prove to be very small at low values of h. A rougher solution (with a larger number of ε) can be found if we weaken the condition of the smallness of disturbances and admit small motions of contact points and small changes in the saturation sections. We will not dwell in detail on this subject.

5. CHECK OF THE PROGRAM FOR OPTIMALITY

Let $v(t)$, $t \in T$, be a control signal at which the terminal constraints and phase constraints are fulfilled accurate to h in the initial problem. Without the loss of generality, we may assume that the appropriate control action $u(t)$, $t \in T$, has one laying section $S_1 = \{s \in T : \ u(s) = L\} = \{s_1, \ldots, s^1\}$. We will verify its optimality in the initial problem (1)–(3), (7)–(9) by solving special OC problems.

First, to check the optimality of the ends of the laying section S_1, we will apply the regularization procedure described in Section 4 to a control action. Then, we will check this section for the completeness by solving the problems of the form (13) on the intervals $[\tau, \tau + (k+1)h]$, $\tau \in \{s_1, s_1 + h, \ldots, s^1 - (k+1)h\}$, with the constraints

$$\begin{gathered} d'y(\tau) = L, d'B^i y(\tau) = 0, \quad i = \overline{1, k-1}; \\ d'y(\tau + (k+1)h) = L, \quad d'B^i y(\tau + (k+1)h) = 0, \quad i = \overline{1, k-1}; \\ |d'B^k y(t)/d'B^{k-1} f| \leq M, \quad t \in [\tau, \tau + (k+1)h]. \end{gathered} \tag{33}$$

If in the solutions of these problems, the control action $u(t)$, $t \in [\tau, \tau + (k+1)h]$, will disturb the direct constraints at any instant from the interval cited above, then the initial control signal is optimal on this interval. Let us note that instead of the problems (13) with constraints (33), we can solve the problems of the form (13) on the intervals $[\tau, \tau + (k+1)h]$, $\tau \in \{s_1, s_1 + h, \ldots, s^1 - (k+1)h\}$, with the constraints

$$\begin{gathered} d'y(\tau) = L, d'B^i y(\tau) = 0, \quad i = \overline{1, k-1}; \\ d'y(\tau + (k+1)h) = L, \quad d'B^i y(\tau + (k+1)h) = 0, \quad i = \overline{1, k-1}; \\ -M \leq -d'B^k y(t)/d'B^{k-1} f \leq 0, \quad t \in [\tau, \tau + (k+1)h]. \end{gathered} \tag{34}$$

In the case when $v(t) = 0$, $t \in [\tau, \tau + (k+1)h]$, is fulfilled in solutions of the problems (13), (34), the initial control signal is optimal on the given interval.

6. OPTIMAL CONTROL IN REAL TIME

As defined in Section 2, the positional solution of the problem (1)–(3), (7)–(9) reduces to the synthesis of optimal feedback (11). It is impossible to produce optimal feedback (11) in the explicit form for any nontrivial problems. Therefore, following [8], we will describe the method of its implementation in each particular process of control. The method relies on the analysis of the use of optimal feedback under real conditions.

We assume that optimal feedback (11) is built up. We will use it for control of a real system, reasoning that the behavior of the system (with due regard for modeling inaccuracies and disturbances) is described by Eq. (12). Let us consider a certain specific process of control in which the initial state x_0^*, y_0^* and the disturbance $w^*(t)$, $t \in T$, were carried out. They will generate in the closed system (12) the process $x^*(t)$, $y^*(t)$, $t \in T$, within which the following identities are met: $\dot{x}^*(t) \equiv Ax^*(t) + bu^*(t) + w^*(t)$, $u^*(t) \equiv d'y^*(t)$, and $\dot{y}^*(t) \equiv By^*(t) + fv^0(t, x^*(t), y^*(t))$, $t \in T$. Hence, it is seen that in the course of control, the input of the system (12) receives the signal

$$v^*(t) = v^0(t, x^*(t), y^*(t)), \quad t \in T, \tag{35}$$

which will be called the implementation of optimal feedback (11) in the process of control. According to (35), to control a particular process, there is no need to know optimal feedback (11) as a whole (in the entire domain of its estimation); it is only necessary to know its values along isolated continuous curves $x^*(t)$, $y^*(t)$, $t \in T$. Moreover, at each current instant $\tau \in T_h$, it is sufficient to know how to calculate the current value of $v^*(\tau) = v^0(\tau, x^*(\tau), y^*(\tau))$ from the measured $x^*(\tau)$ and $y^*(\tau)$ in the time that does not exceed h, i.e., in the real time mode. Thus, the problem of synthesis of optimal feedback (11) reduced to the development of an algorithm for the operation of a device capable of performing this operation [5]. We will call this device a control element.

In the start interval $[0, h]$ when the initial state x_0, y_0 is known, the control element does not actually do anything. It is sufficient for this element to send to the system (1)–(3) a signal $v^*(t) = v^0(0, x_0, y_0) = v^0(0 \mid 0, x_0, y_0)$, $t \in [0, h[$, taken from the optimal program $v^0(t \mid 0, x_0, y_0)$, $t \in T$, which was set up before the beginning of the process of control without any constraints on the calculation time. We will analyze the situation that arises for the control element at an arbitrary current instant $\tau \in T_h$ of the process of control. According to the assumption, the control element already worked out at the preceding instant $\tau - h$ the signal $v^*(\tau - h) = v^0(\tau - h \mid \tau - h, x^*(\tau - h), y^*(\tau - h))$ and knows the optimal programmed control signal $v^0(t \mid \tau - h, x^*(\tau - h), y^*(\tau - h))$, $t \in [\tau - h, t^*]$, and the optimal support $K_{\sup}(\tau - h)$. We denote by $x^0(\tau)$, $y^0(\tau)$ the state of the system into which it would pass at the instant τ from the state $x^*(\tau - h)$, $y^*(\tau - h)$ without a disturbance. The real state $x^*(\tau)$, $y^*(\tau)$ will differ from the ideal state $x^0(\tau)$, $y^0(\tau)$ by the vector $\int\limits_{\tau-h}^{\tau} F(\tau - s)w^*(s)ds$. In the case of a limited disturbance, the quantity $\left\| \int\limits_{\tau-h}^{\tau} F(\tau - s)w^*(s)ds \right\|$ is smaller, the smaller is h. Therefore, the control element described above corrects $K_{\sup}(\tau - h)$ by the dual method until the development of the optimal support $K_{\sup}(\tau)$.

7. DYNAMIC CONTROLLERS WITH STOPS

So far, in this work we examined the dynamic controllers in which the direct constraint on control actions was effected with the aid of control signals. In the text presented below, we investigate an OC problem in which the constraint on a control action of dynamic controllers is afforded in the automatic way on account of the availability of special additional devices (stops). In this situation, control signals only establish optimal transient processes between stops or, in other words, optimally use the stops for the optimization of the entire process of control.

We will restrict ourselves to dynamic controllers of the particular form (6). In view of stops, the OC problem formally has the previous form

$$\begin{aligned}
&c'x(t^*) \to \max_v; \quad \dot{x} = Ax + bu, \quad x(0) = x_0;\\
&u^{(r)} + \alpha_r u^{(r-1)} + \ldots + \alpha_2 \dot{u} + \alpha_1 u = v,\\
&u(0) = u_{00}, \quad \dot{u}(0) = u_{10}, \ldots, u^{r-1}(0) = u_{r-1,0};\\
&Hx(t^*) = g; \quad |v(t)| \le M, \quad |u(t)| \le L, \quad t \in T.
\end{aligned} \tag{36}$$

Changes are involved with the essence of notions. The discrete control signal $v(\cdot) = (v(t),\ t \in T)$ will be called an admissible signal (a program) if there exist instants $0 \le \underline{t}_1 < \overline{t}_1 < \underline{t}_2 < \overline{t}_2 < \ldots < \underline{t}_q < \overline{t}_q \le t^*$ and numbers $\varepsilon_j = \pm 1$, $j = \overline{1,q}$, such that the function $v(t)$, $t \in T$,

(1) takes any values on intervals $[\underline{t}_j, \overline{t}_j]$, $j = \overline{1,q}$,

(2) has a discrete form with the quantization period h and $|v(t)| \le M$ on the set $T \setminus \bigcup_{j=\overline{1,q}} [\underline{t}_j, \overline{t}_j]$,

(3) generates a solution $u_*(t)$, $t \in T$, of an equation of the dynamic controller, such that

$$\begin{gathered}
u_*^{(i)}(0) = u_{i0}, \quad u_*(\underline{t}_j) = u_*(\overline{t}_j) = L \operatorname{sgn} \varepsilon_j,\\
u_*^{(i)}(\overline{t}_j) = 0, \quad i = \overline{1, r-1}; \quad j = \overline{1,q};\\
|u_*(t)| \le L, \quad t \in T \setminus \bigcup_{j=\overline{1,q}} [\underline{t}_j, \overline{t}_j];
\end{gathered}$$

and affords the trajectory $x(t),\ t \in T$, of the system (36) with the control action

$$u(t) = u_*(t), \quad t \in T \setminus \bigcup_{j=\overline{1,q}} [\underline{t}_j, \overline{t}_j]; \quad u(t) = L \operatorname{sgn} \varepsilon_j, \quad t \in [\underline{t}_j, \overline{t}_j], \quad j = \overline{1,q},$$

which satisfies the terminal constraint.

The admissible control signal $v^0(\cdot)$ will be called optimal if on the pertinent (optimal) trajectory $x^0(t)$, $t \in T$, the relation $c'x^0(t^*) = \max c'x(t^*)$ is met, where the maximum is taken over all programs.

Without dwelling on the proof of the criteria of optimality and suboptimality, we will describe for the problem (36) the methods of calculating an optimal program and implementing optimal feedback.

We will calculate an optimal program starting with the solution of an OC problem:

$$\begin{aligned}
&c'x(t^*) \to \max_u; \quad \dot{x} = Ax + bu, \quad x(0) = x_0;\\
&u^{(r)} + \alpha_r u^{(r-1)} + \ldots + \alpha_2 \dot{u} + \alpha_1 u = v,\\
&u(0) = u_{00}, \quad \dot{u}(0) = u_{10}, \ldots, u^{r-1}(0) = u_{r-1,0};\\
&Hx(t^*) = g; \quad |v(t)| \le M, \quad t \in T.
\end{aligned} \tag{37}$$

Let us previously introduce necessary definitions.

Definition 1. The time interval $[\underline{t}_j, \overline{t}_j]$ will be called the section of saturation (of a control action) if[5] (1) $u(t) = L$, $t \in [\underline{t}_j, \overline{t}_j]$; (2) on account of the solution $v(t), u(t)$, $t \in T$, of the extremal problem (37) with the constraints $u(t) = L, t \in [\underline{t}_j, \overline{t}_j]$, and $u(\overline{t}_{j-1}) = L,\ u^{(i)}(\overline{t}_{j-1}) = 0$, $i = \overline{1, r-1}$, the inequalities $u(\underline{t}_j - h) < L$ and $v(\overline{t}_j) < 0$ are met; and (3) on account of the solution $v(t), u(t)$, $t \in T$, of the extremal problem (37) with the constraints $u(t) = L$, $t \in [\underline{t}_j + h, \overline{t}_j - h]$; $u(\overline{t}_{j-1}) = L$, and $u^{(i)}(\overline{t}_{j-1}) = 0$, $i = \overline{1, r-1}$, the inequalities $u(\underline{t}_j) \ge L$, $v(\overline{t}_j - h) \ge 0$ are met.

[5] For definiteness, we further consider only the upper bound of possible values of a control action.

Definition 2. The procedure of regularization of the bedding section $u(t) = L$, $t \in [\underline{t}_j, \overline{t}_j]$, will be called the construction from the last saturation section or its removal. This procedure consists in the following. The bedding section is extended by an elementary interval (of length h) if the appropriate inequality from Condition (2) in Definition 1 is not fulfilled or contracted if Condition (3) in Definition 1 is not fulfilled.

Definition 3. The notion of a point of (approximate) contact (accurate to h) of a control action of the boundary of possible values is introduced in the same way as in Section 4.

Without the loss of generality, it can be assumed that there are no bedding sections on T. We denote by $\underline{t}_1 \in T_h$ the instant that does not define a point of contact and at which an optimal control action of the problem (37) goes out beyond or to the boundary of the phase constraint $|u(\underline{t}_1)| \geq L$ for the first time. Let us assume that $u(\underline{t}_1) \geq L$. We will solve the problem (37) with additional constraints $u(t) = L$, $t \in [\underline{t}_1, \underline{t}_1 + h]$, and $u^{(i)}(\underline{t}_1 + h) = 0$, $i = \overline{1, r-1}$, and perform the correction of the bedding section $[\underline{t}_1, \underline{t}_1 + h]$. We will find a new instant $\underline{t}_2 \in T_h$ that does not specify a point of contact and at which a control action goes out beyond or to the boundary of the phase constraint.

Let us assume that as a result of the performed corrections, we formed a control action and found an instant $\underline{t}_j \in T_h$, such that (1) $u(\underline{t}_j) \geq L$ and (2) on the section $[0, \underline{t}_j]$, the phase constraint is fulfilled, excepting, possibly, the points of contact, and the control $u(t)$, $t \in T$, has saturation sections (to avoid cumbersome calculations, we will consider that there is one such section, i.e., $j = 2$).

We will solve the problem (37) with the additional constraints $u(t) = L$, $t \in [\underline{t}_1, \overline{t}_1]$; $u^{(i)}(\overline{t}_1) = 0$, $i = \overline{1, r-1}$; $u(t) = L$, $t \in [\underline{t}_2, \underline{t}_2 + h]$; and $u^{(i)}(\underline{t}_2 + h) = 0$, $i = \overline{1, r-1}$, and will regularize the bedding sections $[\underline{t}_1, \overline{t}_1]$ and $[\underline{t}_2, \underline{t}_2 + h]$.

Note. The described procedure of the correction will lead, in a finite number of steps, to the development of a control action that everywhere, apart from the finite number of the points of approximate contact, will satisfy the constraint $|u(t)| \leq L$, $t \in T$. It is possible to increase the accuracy of the formation of the phase constraint with the aid of the procedure of sizing [9], which will afford both the exact contact by a control action of the boundaries of possible values in the neighborhood of the points of approximate contact and the fulfillment of the equalities $|u(\underline{t}_j)| = L$, $j = \overline{1, q}$.

8. EXAMPLES

We will first illustrate the obtained results by OC problems considered in [10]. In these problems, the Fuller modes arise at $r \geq 3$.

Example 1. The mathematical model of the problem presented below is given as

$$\begin{aligned} &x(5) \to \min; \quad \dot{x} = u, \quad x(0) = 0; \quad u^{(r)} = v, \\ &u(0) = 1.5, \quad u^{(i)}(0) = 0, \quad i = \overline{1, r-1}; \\ &u(t) \geq 1, \quad |v(t)| \leq 1, \quad t \in T = [0, 5]. \end{aligned} \tag{38}$$

The problem (38) was solved for $r = \overline{2, 4}$. We selected the quantization period $h = 5/50 = 0.1$. In the neighborhood of the points of approximate contact, the net mesh was refined by a factor of 4.

(1) ($r = 2$). In this case, the optimal values of the quality criterion is equal to 6.853938. The saturation section lies in the interval $[1.7; 5]$. The optimal control action and its fragment at a reduced scale are shown in Fig. 1 (curves *1* and *4*).

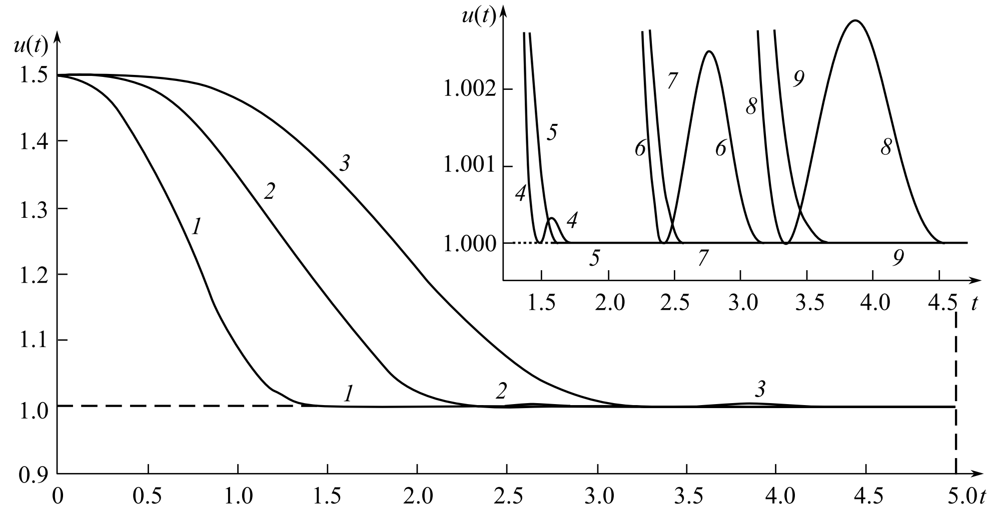

Fig. 1.

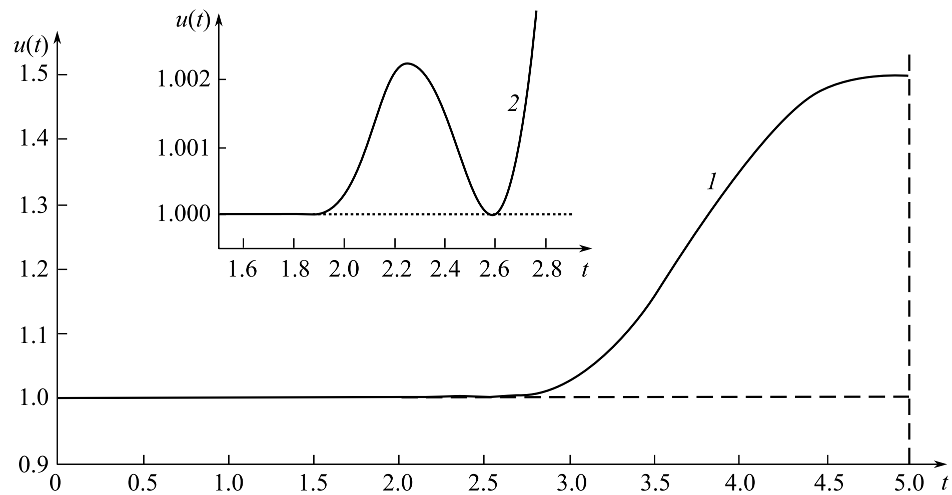

Fig. 2.

For comparison, Fig. 1 also displays a fragment of the optimal control action of the problem (38) (curve *5*), which is built in the following way. A minimum instant $s_* \in \{jh,\ j = \overline{1,\ 49}\}$ is found, at which the problem (38) without the phase constraint and with the condition of the soft going out at the point s_*: $u(s_*) = 1$, $u^{(i)}(s_*) = 0$, $i = \overline{1, r-1}$, has a solution. The obtained solution is extended by an interval $[s_*, 5]$: $u^0(t) = 1$, $t \in [s_*, 5]$. For the control action formed by the described method, the value of the quality criterion is found to be equal to 6.857988, i.e., it is worse than in the preceding case. The saturation section lies in the interval $[1.6; 5]$.

(2) ($r = 3$). The curves *2* and *6* in Fig. 1 illustrate the optimal control action and its fragment, respectively. The value of the quality criterion in this case is equal to 7.132608. The saturation section lies in the interval $[3.2; 5]$. For the optimal control action without oscillations before the going out to the boundary (see the case $r = 2$), the value of the quality criterion is found to be equal to 7.133912. The saturation section lies in the interval $[2.6; 5]$. The fragment of the optimal control action formed in this way is displayed by the curve *7* in Fig. 1.

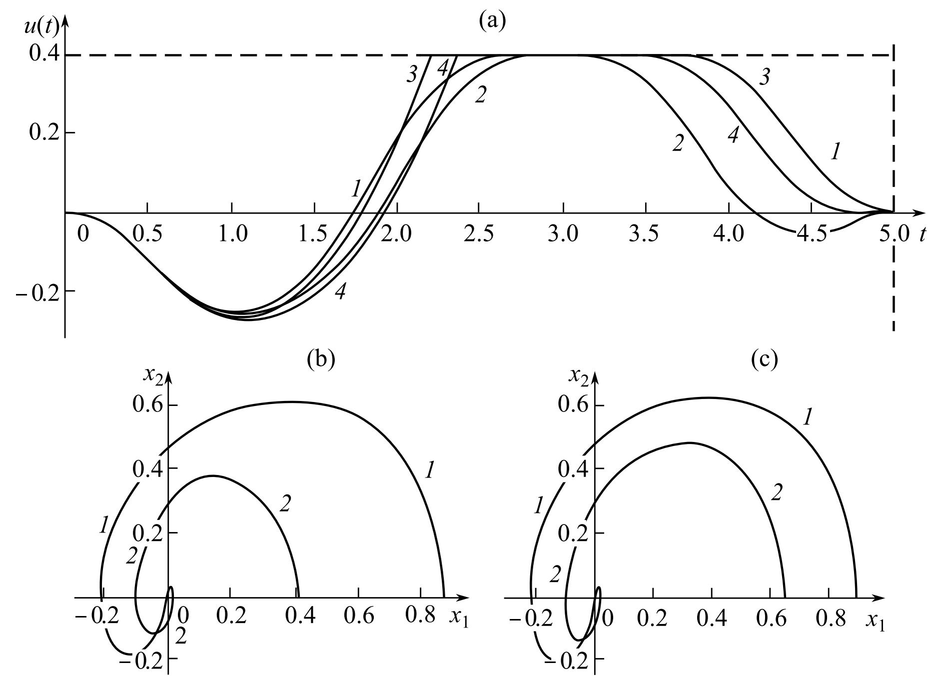

Fig. 3.

(3) ($r = 4$). In this case, the optimal value of the quality criterion is equal to 7.432933. The saturation section lies in the interval $[4.7; 5]$. The curve *3* of Fig. 1 illustrates an optimal control action. Its fragment at a reduced scale and the fragment of the optimal control action of the same type of problem, but without oscillations before the going out to the boundary (see the case $r = 2$), are illustrated in Fig. 1 by the curves *8* and *9*, respectively. In the second case, the value of the quality criterion is found to be equal to 7.436399 and the saturation section proves to be the segment $[3.7; 5]$.

Example 2. We will again solve the OC problem [10] with additional constraints on the right end. We consider the problem

$$\begin{aligned} &x(5) \to \min; \quad \dot{x} = u, \quad x(0) = 0; \quad u^{(3)} = v, \quad u(0) = 1, \quad \dot{u}(0) = 0, \quad \ddot{u}(0) = 0; \\ &u(5) = 1, \quad \dot{u}(5) = 0, \quad \ddot{u}(5) = 0; \quad u(t) \geq 1, \quad |v(t)| \leq 1, \quad t \in T = [0, 5]. \end{aligned} \tag{39}$$

In this case, the optimal value of the quality criterion is equal to 7.132552. The saturation section lies in the interval $[0; 1.8]$. The optimal control action and its fragment at a decreased scale are displayed in Fig. 2.

We will now consider the problem of OC with the aid of the dynamic controller of order 2.

Example 3. The mathematical model of the problem is given as

$$\begin{gathered} x_1(5) \to \max; \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad x_1(0) = 0, \quad x_2(0) = 0; \\ \ddot{u} = v, \quad u(0) = 0, \quad \dot{u}(0) = 0; \quad u(5) = 0, \quad \dot{u}(5) = 0; \\ |u(t)| \leq 0.4, \quad |v(t)| \leq 1, \quad t \in T = [0, 5]. \end{gathered} \tag{40}$$

We selected the quantization period $h = 5/50 = 0.1$. The optimal value of the quality criterion is equal to 0.871406. The saturation section lies in the interval $[2.8; 3.6]$. Figure 3a (curve *1*) illustrates the optimal program control.

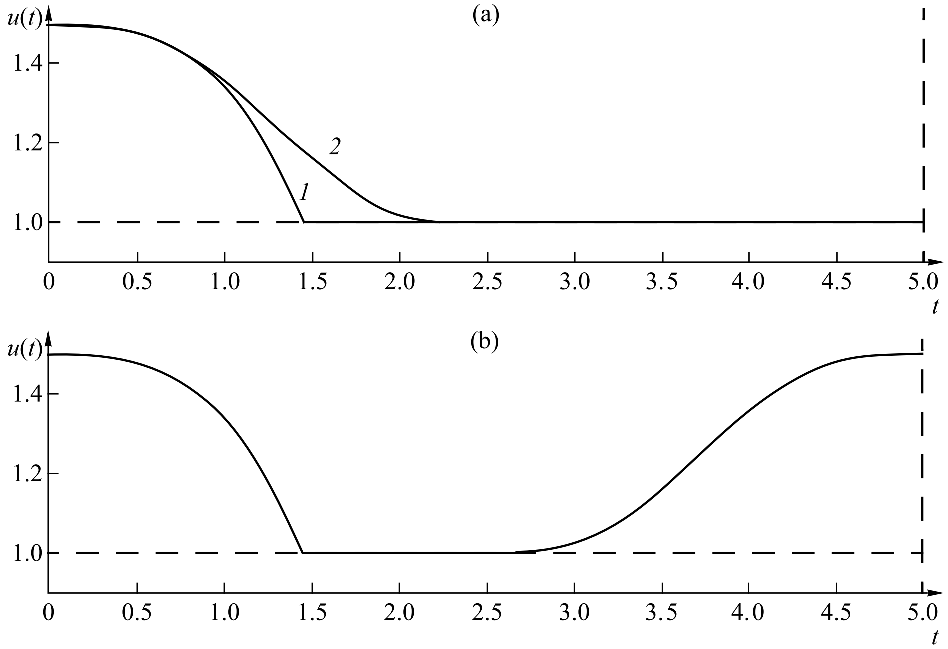

Fig. 4.

For the problem (40), we will implement an optimal positional solution, considering that a limited piecewise-continuous disturbance unknown to the control element acts on the oscillator in the course of control, for which reason the behavior of the oscillator and the dynamic controller is described by the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u + w, \quad \ddot{u} = v. \tag{41}$$

As an implementing disturbance, we take the function $w^*(t) = 0.2 \sin 2t$, $t \in [0; 3.2[$; $w^*(t) = 0$, $t \in [3.2; 5]$. The control element affords the implementation of $v^*(\tau)$, $\tau \in T$, of optimal feedback of the problem (40) relying on the known current states $x^*(\tau)$ and $u^*(\tau)$, $\tau \in T_h$. The curve *2* of Fig. 3a displays the requisite control action $u^*(\tau)$, $\tau \in T$. The optimal value of the quality criterion for the problem (40) is found to be equal to 0.408337. The saturation section lies in the interval $[2.8; 3]$. Figure 3b illustrates trajectories of the systems (4) and (41). The curve *1* displays the optimal program trajectory of the system (40) and the curve *2* illustrates the trajectory of the system (41), which is generated by the control action $u^*(\tau)$, $\tau \in T$, and the disturbance $w^*(\tau)$, $\tau \in T$.

Example 4. We will again solve the problem (38) using the dynamic controller with stops. The optimal value of the quality criterion proves to be equal to 7.040865. The saturation section lies in the interval $[1.442829; 5]$. The curve *1* of Fig. 4a illustrates an optimal control action. For comparison, the curve *2* of Fig. 4a illustrates the optimal control action of the problem (38) with $r = 3$.

Example 5. Using the dynamic controller with stops, we will solve the problem (39) with initial conditions on the control action: $u(0) = 1.5$, $\dot{u}(0) = 0$, $\ddot{u}(0) = 0$ The optimal value of the quality criterion is equal to 7.674864. The saturation section lies in the interval $[1.442829; 5]$. The optimal control action is shown in Fig. 4b.

Example 6. The problem (40) is solved with the use of the dynamic controller with stops. The saturation section lies in the interval $[2.4; 3.6]$. The accuracy of the fulfillment of the phase constraint is increased by means of the sizing procedure that involves a solution of the system of the two equations $u(\tau) = 0.4$ and $\int\limits_0^5 \cos t \; u(t)dt = 0$ with respect to both the instant τ of the going out of a control action to the boundary and a value of the control action at the support instant $t = 0.4$. The optimal value of the quality criterion is equal to 0.8901. The saturation section lies in the interval $[2.205897; 3.6]$. The curve *3* of Fig. 3a illustrates an optimal program control action. We effected the implementation of optimal feedback of the problem (40), which corresponds to the disturbances $w^*(t) = 0.2 \sin 2t$, $t \in [0; 2.6[$; and $w^*(t) = 0$, $t \in [2.6; 5]$. The optimal value of the quality criterion for the problem (41) is equal to 0.647316. The saturation section lies in the interval $[2.36; 3.4]$. Figure 3a displays an optimal control action (curve *4*) that corresponds to optimal feedback. In Fig. 3c are shown trajectories of the systems (40) and (41): the curve *1* represents the optimal program trajectory of the system (40) and the curve *2* represents the trajectory of the system (41).

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