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The Construction of Optimal Feedback from Mathematical Models with Uncertainty

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Abstract—An optimal control problem for dynamical systems with uncertainty is considered. In contrast to the classical approach in which an optimal feedback is constructed on the basis of deterministic models, mathematical models with set-based uncertainty are used. Admissible and optimal controls guarantee a certain result. Depending on available information, several types of feedbacks (unclosable and closable) are introduced. As the feedback becomes more complicated, its effectiveness enhances. For each type of optimal feedback, techniques for its computer implementation are suggested. Results are illustrated by examples.

1. INTRODUCTION

The concept of feedback (which is one of the greatest discoveries of mankind) has been in use for more than 4000 years [1]. It forms the foundation for one of the most effective technique of controlling dynamical systems under uncertainty, which is characteristic of almost all real-life control systems. In the classical period of the development of control theory (up to the 1940s), linear feedbacks were mainly used. In order to improve the quality of control, especially in connection with the first optimal control problems [2, 3], nonlinear feedbacks became popular. A characteristic feature of the classical approach to the synthesis of optimal feedbacks was that they were designed on the basis of deterministic models, even though they were intended for controlling systems under uncertainty. One possible approach to the effective implementation of classical optimal feedbacks is presented in [4].

Engineers realized that taking into account any information about unknown perturbations could enhance the effectiveness of a feedback. However, no appropriate mathematical techniques for such an approach were available until the 1940s. In control theory, information about perturbations was first used in the framework of stochastic models [6, 7] on the basis of the theory of stochastic processes developed at that time [8]. Stochastic models imposed stringent conditions on the necessary information and did not completely represent all specific features of modern models of uncertainty. With time, nonstochastic (set-based) models of uncertainty were developed [9, 10] in which information about perturbations was represented by sets of their possible values. New models of uncertainty made it possible to investigate important applied problems of guaranteed (minimax) control in a natural way; however, they required new mathematical techniques related to nonsmooth analysis. Such techniques were developed in optimal control theory [11, 12] and differential games [13].

It must be noted that the basic results of the new theories were of a qualitative nature; i.e., they enabled one to analyze solutions of the corresponding problems. Although the structure of solutions and potentials of control systems could be investigated, no effective methods for finding the solutions were suggested. The available mathematical methods for solving optimal control problems are mainly aimed at finding open-loop controls. In essence, this is akin to conventional variational calculus and other divisions of mathematics. Problems of controlling dynamical systems are fundamentally different from conventional mathematical problems in that they require the construction of closed-loop solutions; i.e., controls must be calculated in the process of control depending on situations that could not be predicted in advance. In studies devoted to the investigation of open-loop solutions, this specific feature was actually ignored. One well-known exception is provided by linear–quadratic Letov–Kalman problems and the related H_∞-theory. However, these problems are very specific and essentially belong to classical variational calculus. They do not take into account direct (geometric) constraints imposed on controls, which makes it possible to obtain a solution in the form of a linear feedback. Now, after the fifty-year long development of optimal control theory, it is time to remove the restriction of linear feedbacks and proceed to solving optimal control problems on the basis of the first engineering statements [2, 3].

The causes of difficulties arising when the classical results (obtained by Pontryagin and Bellman) are used to synthesize an optimal feedback can be explained as follows. These results are based on the use of adjoint and Bellman equations. For each of these equations, the “natural” time is reversed; i.e., they are normally solved in the time that passes in the reverse direction relative to the real time. This circumstance is of no importance when open-loop solutions are constructed, since such solutions are calculated beforehand rather than in the process of actual control. However, if controls must be constructed in real time (in particular, if a feedback is used), the time involved in the adjoint equation or the Bellman equation must be reversed. The classical method of reversing time is to find explicit formulas for the feedback (optimal control) for all possible states of the system. Another method is to store (tabulate) the optimal control values for all possible states of the system. Both methods cannot be implemented in practice using available mathematical techniques and computer resources due to the high complexity of real-life systems and the curse of dimensionality. The approach to the synthesis of optimal feedbacks suggested in this paper can be interpreted as a technique for reversing time in the course of actual control based on constant solution of special extremal problems. The result can be called the optimal closing of the control system for states that occur in the course of actual control. From this point of view, the optimal open-loop control closes the control system before the start of the actual control for the initial state; this makes it impossible to take into account perturbations that affect future states of the system. An optimal feedback constructed using the maximum principle or dynamic programming closes the system before the start of the actual control for all possible states of the system (including those that will never occur in the course of actual control).

The purpose of this paper is to generalize the results on the synthesis of classical optimal feedbacks obtained in [4, 14] to the case when the mathematical model of the control systems involves set-based uncertainties. A similar problem was studied in [15]; however, the techniques used in that paper are difficult to extend to higher order systems. The approach used in this paper follows the basic idea underlying all our studies on the synthesis of optimal systems. Namely, before the actual control process starts, some preliminary work on obtaining a scheme (or skeleton) of a solution is done; then, in the course of actual control, the preliminary solution is constantly refined on the basis of information about the states of the system that are actually realized in the course of control. No effort and resources are spent to process and store information about other possible but not realized states. Ultimately, this approach enables us to work around the difficulties of the classical approach to constructing an optimal feedback in a closed form and the curse of dimensionality, which is characteristic of dynamic programming. Moreover, this approach can be applied not only to deterministic optimal control problems considered in [4, 14], but also to much more complicated problems that actually belong to the theory of differential games.

The paper is organized as follows. In Section 2, a statement of the problem on the synthesis of optimal systems from a nondeterministic model is given. In contrast to the classical case, the linear model under consideration involves an additive unknown perturbation function. This function is introduced to take into account inaccuracies of the mathematical model and unknown perturbations that occur in the course of actual control. The information about the perturbation is represented in the form of the set of its possible values. In this situation, it is natural to consider the optimal guaranteed control problem, which can be interpreted as a problem in differential game theory. A dynamic programming method for constructing the optimal feedback is described, and the curse of dimensionality, which makes this method impossible to implement, is illustrated. In Section 3, the concept of optimal unclosable feedback is introduced, and a practical method for its construction is suggested. The capabilities of the optimal unclosable feedback are extended in Section 4, where we introduce an initial stage that enables one to start the process of control when no optimal unclosable feedback exists for the initial state. Beginning with Section 5, optimal closable feedbacks are investigated. When constructing such a feedback, not only the fact that the feedback is closed at the current state is used but also the fact that it will be closed in the future. First, the case of a single closing instance is investigated. In Section 7, a method for constructing an optimal feedback for several closing instances is presented. The optimal closable feedback in which the set of closing instances coincides with the set of all instances at which a control can be applied is called the optimal closed feedback. The implementation of such a feedback does not assume the tabulation of functions of several variables and, therefore, can be carried out on modern computers. The results are illustrated by examples in Sections 6 and 8.

2. STATEMENT OF THE PROBLEM

Consider the optimal control problem

$$\begin{gathered} h'_{(0)}x(t^*) \longrightarrow \max, \quad \dot{x} = A(t)x + b(t)u + d(t)w, \quad x(t_*) = x_0, \\ x(t^*) \in X^* = \{x \in \mathbb{R}^n : h'_{(i)}x \ge g_i,\ i = \overline{1, m}\}, \\ |u(t)| \le 1, \quad w(t) \in W = \{w \in \mathbb{R} : w_* \le w \le w^*\}, \quad t \in T = [t_*, t^*], \end{gathered} \tag{2.1}$$

where $x = x(t)$ is the n-dimensional state vector of the dynamical system at the instance t, $u = u(t)$ is the scalar control, and $w = w(t)$ $(t \in T)$ is a piecewise continuous perturbation function.

The function

$$u = u(t, x), \quad x \in X_t, \quad t \in T, \tag{2.2}$$

with the domains $X_t (t \in T)$ is called the feedback in problem (2.1).

Let us close system (2.1) using this feedback:

$$\dot{x} = A(t)x + b(t)u(t, x) + d(t)w, \quad x(t_*) = x_0. \tag{2.3}$$

We assume that, for every possible perturbation $w(\cdot) = (w(t), t \in T)$, there exists a unique solution $x(t \mid w(\cdot))$ $(t \in T)$ to Eq. (2.3). Let $X(t) = \{x : x = x(t \mid w_t(\cdot))$, where $w(s) \in W$, $t_* \le s \le t\}$, $w_t(\cdot) = (w(s), t_* \le s \le t)$.

Feedback (2.2) is called admissible if

$$|u(t, x)| \le 1, \quad x \in X_t, \quad t \in T,$$

and the final state of system (2.3) at $t = t^*$ belongs to the terminal set $X^* : X(t^*) \subset X^*$ for any possible perturbation.

The quality of the feedback is assessed by the value

$$J(u) = \min h'_{(0)}x, \quad x \in X(t^*).$$

An admissible feedback $u^0 = u^0(t, x)$ $(x \in X_t, t \in T)$ is called optimal if $J(u^0) = \max J(u)$, where the maximum is taken over all admissible feedbacks.

Dynamic programming [12] provides a universal method for constructing optimal feedbacks. In order to avoid issues related to the justification of dynamic programming for general feedbacks (2.2), we introduce the concept of a discrete feedback that differs from (2.2) only in the way it affects the dynamical system (2.3). Let $h = (t^* - t_*)/N$ $(N > 0)$ be the quantization interval. A feedback (2.2) is called discrete with the quantization interval h if the trajectory of system (2.3) closed by this feedback coincides with the trajectory of the linear equation

$$\dot{x} = A(t)x + b(t)u(t) + d(t)w, \quad x(t_*) = x_0,$$

with the control $u(t) = u(t_* + kh, x(t_* + kh))$ $(t \in [t_* + kh, t_* + (k+1)h[$, $k = \overline{0, N-1}$).

It is clear that in the case of a discrete feedback, problems of definition and existence of solutions to Eq. (2.3) with a discontinuous (in x) right-hand side do not arise.

Discrete feedbacks determine discrete (open-loop) controls

$$u(t) = u(t_* + kh), \quad t \in [t_* + kh, t_* + (k+1)h[, \quad k = \overline{0, N-1}. \tag{2.4}$$

We embed problem (2.1) considered in the class of open-loop controls in the family

$$\begin{gathered} B(\tau, z) = \max_u \min_w h'_{(0)}x(t^*), \quad \dot{x} = A(t)x + b(t)u + d(t)w, \quad x(\tau) = z, \\ x(t^*) \in X^*, \quad |u(t)| \le 1, \quad w(t) \in W, \quad t \in T(\tau) = [\tau, t^*], \end{gathered} \tag{2.5}$$

depending on the scalar $\tau \in T_h = \{t_*, t_* + h, \ldots, t^* - h\}$ and the n-dimensional vector $z \in X_\tau$.

Using the standard reasoning that is common for dynamic programming, we obtain the following recurrence for the Bellman function $B(\tau, z)$ $(z \in X_\tau, \tau \in T_h)$:

$$B(\tau, z) = \max_{|v| \le 1} \min_{w(s) \in W,\, s \in [\tau, \tau + h[} B(\tau + h, x(\tau + h \mid \tau, z, v, w_h(\tau))). \tag{2.6}$$

Here, $x(\tau + h \mid \tau, z, v, w_h(\tau))$ is the state of system (2.5) to which it goes at the instance $\tau + h$ from the state z at the instance τ under the influence of the control $u(t) = v$ $(t \in [\tau, \tau + h[)$ and the perturbation $w_h(\tau) = (w(t), t \in [\tau, \tau + h[)$.

The initial condition for the recurrent equation (2.6) is

$$B(t^*, z) = \begin{cases} h'_{(0)}z, & z \in X^*, \\ -\infty, & z \notin X^*. \end{cases}$$

Equation (2.6) can be theoretically solved step-by-step from the right to the left, and incidentally the optimal discrete feedback $u^0(t, x)$, $x \in X_t$, $t \in T_h$ $(X_t = \{x \in \mathbb{R}^n : B(\tau, x) > -\infty\})$ is constructed on the basis of the equation

$$\begin{aligned} &\min_{w(s) \in W,\, s \in [\tau, \tau+h[} B(\tau + h, x(\tau + h \mid \tau, z, u^0(\tau, z), w_h(\tau))) \\ &= \max_{|u| \le 1} \min_{w(s) \in W,\, s \in [\tau, \tau+h[} B(\tau + h, x(\tau + h \mid \tau, z, u, w_h(\tau))). \end{aligned}$$

This elegant scheme of solving problem (2.1) by the dynamic programming method is impracticable due to the curse of dimensionality (see [12]), i.e., due to the impossibility to tabulate the Bellman function and feedback for problem (2.1) when n is large.

For this reason, other types of optimal feedbacks were suggested in [15] that are simpler than those obtained by the dynamic programming method (however, they can be less efficient). Simplified feedbacks for the network extremal problem with a stochastic model of uncertainty were first described in [16]. The purpose of this paper is to present a new (compared with [15]) method of implementing feedback controls for problem (2.1), including the optimal feedback obtained by the dynamic programming method.

3. OPTIMAL UNCLOSABLE FEEDBACK

Consider problem (2.1) in the class of discrete controls (2.4). Embed it in the family of problems (2.5), and denote by $u^0(t \mid \tau, z)$ $(t \in T(\tau))$ the open-loop solution for the state (τ, z). Let X_τ be the set of $z \in \mathbb{R}^n$ for which problem (2.5) has a solution.

The function

$$u^0(\tau, z) = u^0(\tau \mid \tau, z), \quad z \in X_\tau, \quad \tau \in T_h, \tag{3.1}$$

is called the optimal unclosable discrete feedback.

Before describing an algorithm for the implementation of feedback (3.1), we consider the problem of constructing the optimal open-loop control $u^0(t \mid \tau, z)$ $(t \in T, (\tau))$.

The discrete control $u(t)$ $(|u(t)| \le 1, t \in T(\tau))$ is admissible if and only if, for all $w(t) \in W$ $(t \in T(\tau))$, the corresponding trajectories

$$x(t) = F(t)F^{-1}(\tau)z + \int_\tau^t F(t)F^{-1}(s)[b(s)u(s) + d(s)w(s)]ds,$$

$$\dot{F} = A(t)F, \quad F(t_*) = E,$$

fall into the terminal set X^* at the instance t^*, i.e., when

$$h'_{(i)}x(t^*) \ge g_i, \quad i = \overline{1, m}.$$

Obviously, this condition is satisfied if

$$h'_{(i)}F(t^*)F^{-1}(\tau)z + \int_\tau^{t^*} h'_{(i)}F(t^*)F^{-1}(s)b(s)u(s)ds \ge g_i - \gamma_i(\tau), \tag{3.2}$$

where

$$\gamma_i(\tau) = \min_{w(t) \in W,\, t \in T(\tau)} \int_{\tau}^{t^*} h'_{(i)} F(t^*) F^{-1}(s) d(s) w(s) ds.$$

Therefore, Eqs. (3.2), $|u(t)| \le 1$ ($t \in T(\tau)$) describe admissible discrete controls $u(t)$ ($t \in T(\tau)$).

By analogy with (3.2), we evaluate the quality of an admissible control as

$$\begin{aligned} J(u) &= h'_{(0)} F(t^*) F^{-1}(\tau) z + \int_{\tau}^{t^*} h'_{(0)} F(t^*) F^{-1}(s) b(s) u(s) ds \\ &+ \min_{w(t) \in W,\, t \in T(\tau)} \int_{\tau}^{t^*} h'_{(0)} F(t^*) F^{-1}(s) d(s) w(s) ds. \end{aligned} \tag{3.3}$$

Hence, it follows that the optimal open-loop control in problem (2.5) under uncertainty conditions is a solution to the deterministic extremal problem

$$\begin{gathered} \int_{\tau}^{t^*} h'_{(0)} F(t^*) F^{-1}(s) b(s) u(s) ds \longrightarrow \max, \\ h'_{(i)} F(t^*) F^{-1}(\tau) z + \int_{\tau}^{t^*} h'_{(i)} F(t^*) F^{-1}(s) b(s) u(s) ds \ge g_i - \gamma_i(\tau), \quad i = \overline{1, m}, \\ |u(t)| \le 1, \quad t \in T(\tau). \end{gathered} \tag{3.4}$$

The dynamic form of problem (3.4) is

$$\begin{gathered} h'_{(0)} x(t^*) \longrightarrow \max, \quad \dot{x} = A(t) x + b(t) u, \quad x(\tau) = 0, \\ x(t^*) \in \bar{X}^*(\tau, z) = \{x \in \mathbb{R}^n : h'_{(i)} x \ge \bar{g}_i(\tau),\, i = \overline{1, m}\}, \quad |u(t)| \le 1, \quad t \in T(\tau), \end{gathered} \tag{3.5}$$

where $\bar{g}_i(\tau) = g_i - \gamma_i(\tau) - h'_{(i)} F(t^*) F^{-1}(\tau) z$ ($i = \overline{1, m}$).

An algorithm for constructing open-loop solutions $u^0(t \mid \tau, z)$ ($t \in T(\tau)$) in problems (3.5) is described in [14]. It is a dynamic implementation of the adaptive linear programming method presented in [17] that takes into account specific features of optimal control problems.

In order to define the concept of realization of the optimal feedback (2.2), we analyze the behavior of the closed system (2.3) in a particular control process.

Assume that the initial state of the system is x_0^* and the perturbation $w^*(t)$ ($t \in T$) is realized in the process under consideration. This initial state and the perturbation induce the transient process $x^*(t)$ ($t \in T$) in system (2.3), along which the identity

$$\dot{x}^*(t) \equiv A(t) x^*(t) + b(t) u^0(t, x^*(t)) + d w^*(t), \quad t \in T, \quad x(t_*) = x_0^* \tag{3.6}$$

holds. It is seen from (3.6) that feedback (2.2) is not completely used in this particular control process; only its values

$$u^*(t) = u^0(t, x^*(t)), \quad t \in T_h, \tag{3.7}$$

along the curve $x^*(t)$ ($t \in T$) are important. It is sufficient to know how to calculate the value $u^*(\tau) = u^0(\tau, x^*(\tau))$ at the current time $\tau \in T_h$ from the current state $x^*(\tau)$ in time not exceeding h, i.e., in real time.

Function (3.7) is called the realization of the optimal feedback in the particular control process; a device that calculates the values of this function in real time for a particular control process (3.6) is called the optimal controller. The optimal controller works according to the following algorithm. Before the initial time $\tau = t_*$, the controller solves problem (3.5) for $\tau = t_*$, $z = x_0^*$, and feeds the signal $u^*(t_*) = u^0(t_* \mid t_*, x_0^*)$ to the entry of system (2.3). Since the calculations are performed before the start of the control process, one can assume that there is enough time to construct the open-loop control of problem (3.5) and acquire auxiliary information as described in [14].

Then, at every current instance $\tau \in T_h$, the controller constructs an open-loop solution to problem (3.5) for the state $(\tau, x^*(\tau))$ using the optimal elements constructed at the preceding instance $\tau - h$ as the initial approximation for the associated elements of the optimal control $u^0(t \mid \tau, x^*(\tau))$ $(t \in T(\tau))$ (see [14]). In [14], estimates of the computational cost of the method were obtained that make it possible to formulate requirements for the performance of a processor on which the optimal controller is implemented so that the computation time of $u^*(\tau)$ does not exceed h time units. These requirements can be significantly relaxed if several processors (not necessarily high-performance ones) are used to implement feedback (3.1). Although this involves latency in the transmission of signals, the control process can be organized in such a way that the influence of latency is negligible [18].

4. TWO-PHASE OPTIMAL UNCLOSABLE FEEDBACK

It follows from the definition of the optimal unclosable feedback (3.1) that its realization $u^*(\tau)$ $(\tau \in T)$ in a particular control process can be constructed if the initial state of the system satisfies the condition $x_0^* \in X_{t_*}$.

If this condition does not hold, one cannot guarantee a priori that the terminal conditions of problem (2.1) will be satisfied for any possible perturbation. In this case, it is reasonable to organize the control of the system at the initial phase in such a way that the system is brought into the set $X_{\bar{t}}$ at a certain instance $\bar{t} \in T_h$; then, one can proceed to the construction of a realization of the optimal feedback by solving problem (3.5) (the second phase). This scheme implements the two-phase optimal unclosable feedback.

The aim of the first phase problem is to bring the terminal state of system (2.5) into the minimal γ-neighborhood of the terminal set X^*:

$$\begin{gathered}\gamma^0(\tau, z) = \min \gamma, \quad \dot{x} = A(t)x + b(t)u, \quad x(\tau) = 0, \\ h'_{(i)}x(t^*) \ge \bar{g}_i(\tau) - \gamma, \quad i = \overline{1, m}, \quad |u(t)| \le 1, \quad t \in T(\tau).\end{gathered} \tag{4.1}$$

An optimal open-loop control $u_\gamma^0(t \mid \tau, z)$ $(t \in T(\tau))$ in problem (4.1) exists for any $z \in \mathbb{R}^n$.

The two-phase optimal unclosable feedback is determined by the relation

$$u^0(\tau, z) = \begin{cases} u_\gamma^0(\tau \mid \tau, z), & z \notin X_\tau, \\ u^0(\tau \mid \tau, z), & z \in X_\tau, \quad \tau \in T_h. \end{cases}$$

The condition $z \in X_\tau$ is equivalent to the inequality $\gamma^0(\tau, z) \le 0$. Therefore, if problem (3.5) has no solutions for the state (t_*, x_0^*), then the controller constructs solutions to problem (4.1) along the trajectory $x^*(\tau)$ $(\tau \ge t_*)$ being realized while the condition $\gamma^0(\tau, x^*(\tau)) > 0$ is satisfied; it feeds the control $u^*(\tau) = u_\gamma^0(\tau \mid \tau, x^*(\tau))$ to the entry of the system.

If the inequality $\gamma^0(\tau, x^*(\tau)) > 0$ holds for the entire T_h, then the trajectory of the system reaches the $\gamma^0(t^* - h, x^*(t^* - h))$-neighborhood of the set X^* at the instance t^*.

If there exists a $\bar{t} \le t^* - h$ such that $\gamma^0(\bar{t}, x^*(\bar{t})) \le 0$, the controller proceeds to the second control phase beginning at the instance $\bar{t}$; at this phase, the realization $u^*(\tau)$ $(\tau \ge \bar{t})$ is constructed by solving problem (3.5).

In order to switch from the first phase to the second one (i.e., produce the control $u^*(\bar{t})$) in real time or, in other words, avoid latency when switching from solving problem (4.1) to solving problem (3.5), it is sufficient to prepare an approximation to the solution of problem (3.5) by the time $\bar{t}$. To this end, at every $\tau = t_*, \ldots, \bar{t} - h$, after computing $\gamma^0(\tau, x^*(\tau))$ for the current state $(\tau, x^*(\tau))$, we substitute these data into the problem

$$\begin{gathered}h'_{(i)}x(t^*) \longrightarrow \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(\tau) = 0, \\ h'_{(i)}x(t^*) \ge \bar{g}_i(\tau) - \gamma^0(\tau, z) - \varepsilon, \quad |u(t)| \le 1, \quad t \in T(\tau).\end{gathered} \tag{4.2}$$

Here, $\varepsilon > 0$ is a small parameter that ensures, on the one hand, the nonuniqueness of the admissible control in problem (4.2), and the closeness of open-loop controls in problems (4.2) and (3.5) for the state $(\bar{t}, x^*(\bar{t}))$ on the other hand. The parameter ε is chosen so that the construction of the optimal control in problem (3.5) for the state $(\bar{t} - h, x^*(\bar{t} - h))$ by the dual method [14] does not take more than h time units given the associated elements of the optimal control in problem (4.2) for the state $(\bar{t}, x^*(\bar{t}))$.

5. OPTIMAL CLOSABLE FEEDBACK WITH A SINGLE CLOSING INSTANCE OF TIME

When implementing the unclosable feedback (Section 3), the control system (2.1) was closed at each current instance $\tau \in T_h$; however, the information about the fact that the system can be closed at certain future instances was not used. In the case of deterministic models (see [4, 14]), this additional information does not improve the (classical) optimal feedback. However, in the case of models with uncertainty, the use of this kind of additional information can significantly improve the effectiveness of the feedback.

First, consider the simplest case. Assume that before the start of the control process it is known that the control system will be closed at the instance $t^1 \in T_h$. This means that, in order to satisfy the terminal conditions, the trajectory of the system must fall into the set X^1 at the instance t^1; for every point in this set, there exists an admissible control that takes the system into the set X^* at the instance t^* for any perturbation. We assume that the available hardware and software can process this additional information along with the basic information acquired in the process of control. As in Section 3, we assume that the current state $x^*(\tau)$ is known at each current instance $\tau \in T_h$.

5.1. Optimal Open-Loop Control

First, we present an algorithm for constructing the optimal open-loop control in problem (2.5) with a single closing instance. To this end, we introduce some auxiliary constructs.

Take an instance $t^1 \in T_h$ $(t_* < t^1 < t^* - h)$ and assume that $\tau \in T_h^0 = \{t_*, \ldots, t^1 - h\}$ in problem (2.5). Using the closing instance t^1, we define the closing set X^1 consisting of all initial states x at the instance t^1 that can be surely taken to the terminal set X^* by the instance t^*.

Calculate

$$\alpha_{\min} = \min_{x \in X^*} h'_{(0)}x, \quad \alpha_{\max} = \max_{x \in X^*} h'_{(0)}x.$$

For any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, we define the set

$$X^*_\alpha = X^* \cap \{x \in \mathbb{R}^n : h'_{(0)}x \geq \alpha\}.$$

The discrete control $u(t)$ $(t \in T^1 = [t^1, t^*[)$ is called available on the interval T^1 if it satisfies the inequality $|u(t)| \leq 1$ $(t \in T^1)$.

The set $X^1_\alpha \subset \mathbb{R}^n$ is called the α-set of closing at the instance t^1 if, for any $x \in X^1_\alpha$, there exists an available control $u^\alpha(t \,|\, t^1, x, 1)$ $(t \in T^1)$ on the interval T^1 that takes the trajectory of system (2.5) from the state $x(t^1) = x$ to the set X^*_α by the instance t^* for any realization of the perturbation $w(t) \in W$ $(t \in T^1)$. The control $u^\alpha(t \,|\, t^1, x, 1)$ $(t \in T^1)$ is called α-admissible for the state (t^1, x).

In order to determine the set X^1_α, we use the deterministic model

$$\dot{x} = A(t)x + b(t)u, \tag{5.1}$$

and define the restriction of X^*_α in which the worst possible effect of perturbations on the interval T^1 is taken into account:

$$\bar{X}^*_\alpha = \{x \in \mathbb{R}^n : h'_{(i)}x \geq g_i - \gamma_i(t^1),\ i = \overline{1, m};\ h'_{(0)}x \geq \alpha - \gamma_0(t^1)\}.$$

Now, we define X^1_α as the set of states $x \in \mathbb{R}^n$ such that for each of them there exists an available control on the interval T^1 that takes the trajectory of system (5.1) from the state $x(t^1) = x$ to the set $\bar{X}^*_\alpha$ by the instance t^*.

Thus, according to the definition, if the trajectory of system (2.5) falls into the set X_α^1 at t^1 from the initial state $x(\tau) = z$, then an appropriate control on the interval T^1 can guarantee the value of the objective function (2.5) not less than α.

The available control $u^\alpha(t \mid \tau, z, 0)$ $(t \in T^0(\tau) = [\tau, t^1[)$ is called α-admissible for the state (τ, z) if, for all realizations of the perturbation $w(t) \in W$, $t \in T^0(\tau)$, it takes the trajectory of system (2.5) from the state $x(\tau) = z$ to the set X_α^1 by the instance t^1.

The maximum value α^0 for which there exists the set $X_{\alpha^0}^1$ and an α^0-admissible control $u^{\alpha^0}(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$ is the optimal guaranteed value of the objective function of problem (2.5) for the closing instance t^1.

The set of states that can be occupied by system (2.5), under the effect of the control $u^\alpha(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$ and any realization of the perturbation $w(t) \in W$ $(t \in T^0(\tau))$, is denoted by $X_\alpha^1(\tau, z)$. The optimal open-loop control in problem (2.5) with the closing instance t^1 is the α^0-admissible control $u^{\alpha^0}(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$ and the family of α^0-admissible controls $u^{\alpha^0}(t \mid t^1, x, 1)$ $(t \in T^1, x \in X_{\alpha^0}^1(\tau, z))$.

Now, we describe an algorithm for the evaluation of α^0 and construction of $u^{\alpha^0}(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$.

In the general case, the sets X_α^1 $(\alpha \in [\alpha_{\min}, \alpha_{\max}])$ have a very complicated structure, and we do not try to give an exact description of those sets. Instead, since we deal with an application problem, we describe a technique that yields a solution within a certain predefined accuracy. To this end, take an initial $\alpha \in [\alpha_{\min}, \alpha_{\max}]$; from $m + 1$ faces of the set X_α^* find the normal vectors to the "frame" faces of the polyhedron $X_\alpha^{1,1}$, which approximates the set X_α^1; and write equations for the faces of this polyhedron. Then, we construct the sequence $\alpha_1, \dots, \alpha_{k^*}$ converging to α^0 and the sequence of polyhedra $X_{\alpha_k}^{1,k}$ $(k = \overline{1, k^*})$ that refines the structure of the set $X_{\alpha_k}^1$ in the neighborhood of the set $X_{\alpha_k}^1(\tau, z)$ of possible states of system (2.5) at the instance t^1.

Set $k = 1$ and $M(k) = m$.

One iteration of the algorithm involves two procedures: (1) computation of α_k and the corresponding suboptimal control on the interval $T^0(\tau)$ for the given number $M(k) + 1$ of the faces of the approximating polyhedron $X_\alpha^{1,k}$; (2) increasing the number of faces of the approximating polyhedron and finding the normal vectors to the new faces.

The first procedure begins with the construction of the approximating polyhedron $X_\alpha^{1,k}$ for a certain $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. It is constructed on the basis of solving $M(k) + 1$ optimal control problems

$$\begin{gathered}\beta_i^{1\alpha} = \min_{y, u} h_{(i)}^{1\prime} y, \quad \dot{x} = A(t)x + b(t)u, \quad x(t^1) = y, \\ x(t^*) \in \bar{X}_\alpha^*, \quad |u(t)| \le 1, \quad t \in T^1, \quad i = \overline{0, M(k)},\end{gathered} \tag{5.2}$$

where $h_{(i)}^{1\prime} = h'_{(i)} F(t^* - t^1)/\|F'(t^* - t^1)h_{(i)}\|_2$ $(i = \overline{0, m})$. As a result, we obtain the set

$$X_\alpha^{1,k} = \{x \in \mathbb{R}^n : h_{(i)}^{1\prime} x \ge \beta_i^{1\alpha}, i = \overline{0, M(k)}\}.$$

The control $u^{\alpha,k}(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$ is called αk-subadmissible for the state (τ, z) if $X_\alpha^1(\tau, z) \subset X_\alpha^{1,k}$; and the $\alpha_k k$-subadmissible control $u^{\alpha_k, k}(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$ with the maximal value $\alpha = \alpha_k$ is called k-suboptimal control in problem (2.5) for the state (τ, z).

In order to pass to a determinate system, we define the restriction $\bar{X}_\alpha^{1,k}$ of the set $X_\alpha^{1,k}$ as

$$\bar{X}_\alpha^{1,k} = \{x \in \mathbb{R}^n : h_{(i)}^{1\prime} x \ge \bar{\beta}_i^{1\alpha}, i = \overline{0, M(k)}\},$$

where

$$\bar{\beta}_i^{1\alpha} = \beta_i^{1\alpha} - \min_{w(t) \in W,\, t \in T^0(\tau)} \int_\tau^{t^1} h_{(i)}^{1'} F(t^1 - s) d(s) w(s) ds.$$

The problem of finding the k-suboptimal control $u^{\alpha_k, k}(t \mid \tau, z, 0)$ ($t \in T^0(\tau)$) belongs to the family of optimal control problems

$$\begin{gathered} \rho_\alpha = \min_u \rho, \quad \dot{x} = A(t)x + b(t)u, \quad x(\tau) = z, \\ x(t^1) \in \bar{X}_\alpha^{1,k}, \quad |u(t)| \le \rho, \quad t \in T^0(\tau), \end{gathered} \tag{5.3}$$

depending on the parameter $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ at $\alpha = \alpha_k$.

A discrete control $u_\alpha(t)$ ($t \in T^0(\tau)$) in problem (5.3) for a fixed α is admissible if it ensures that the terminal condition $x(t^1) \in \bar{X}_\alpha^{1,k}$ is satisfied; if this control also minimizes the objective function, it is called optimal and is denoted by $u_\alpha^0(t)$ ($t \in T^0(\tau)$).

The optimal control $u_{\alpha_k}^0(t)$ ($t \in T^0(\tau)$) in problem (5.3) corresponding to $\alpha = \alpha_k$ at which $\rho_{\alpha_k} = 1$ is a k-suboptimal control $u^{\alpha_k, k}(t \mid \tau, z, 0)$ ($t \in T^0(\tau)$) in problem (2.5).

Therefore, we can use the following algorithm for constructing a k-suboptimal control in problem (2.5).

Algorithm 1

Step 1. Set $\bar{\rho} = 0$.

Step 2. Solve problem (5.2) $M(k) + 1$ times.

Step 3. Form the set $\bar{X}_\alpha^{1,k}$.

Step 4. If $\bar{X}_\alpha^{1,k} = \varnothing$, then check whether it is possible to make $\bar{X}_\alpha^{1,k} \ne \varnothing$ by decreasing α. If this is possible, we set $\alpha_{\max} = \alpha$, decrease α, and go to Step 2. Otherwise, there is no guaranteed control for the closing instance t^1 in problem (2.5).

Step 5. Solve problem (5.3) to find ρ_α.

Step 6. If $\alpha = \alpha_{\max}$ and $\rho_\alpha \le 1$ or $1 - \varepsilon \le \rho_\alpha \le 1$ (where ε is the given accuracy), then we complete the solution and set $\alpha_k = \alpha$ and $u^{\alpha_k, k}(t \mid \tau, z, 0) = u_{\alpha_k}^0(t)$ ($t \in T^0(\tau)$).

If $\rho_\alpha = \bar{\rho} > 1$, then there are no admissible solutions in problem (2.5).

If $\rho_\alpha > 1$, we decrease α. If $\rho_\alpha < 1 - \varepsilon$, we increase α. Set $\bar{\rho} = \rho_\alpha$ and go to Step 2.

The increment of α is decreased as ρ_α approaches unity.

Problems (5.2) are solved concurrently using a simple modification of the fast algorithm designed for solving the linear nonstationary optimal control problem described in [14]. The value of α varies insignificantly when problems (5.2) are solved at the next step. Therefore, the numbers β_i^α ($i = \overline{0, M(k)}$) also vary insignificantly, and storing the auxiliary information (see [14]) helps find the optimal solution for the next value of α very quickly.

The same fast algorithm described in [14] is also used to solve problem (5.3), which is reduced to the form studied in [14] by the change of variables $\xi_0 = 1/\rho$, $\xi(t) = u(t)/\rho$ ($t \in T^0(\tau)$). Again, the auxiliary information stored for the current α significantly reduces the computation time required to solve problem (5.3) for the next value of α.

Assume that α_k and the k-suboptimal control $u^{\alpha_k, k}(t \mid \tau, z, 0)$ ($t \in T^0(\tau)$) are found, and describe the second procedure of the algorithm.

Denote by $x^{\alpha_k}(t^1)$ the point of the trajectory of system (5.1) at the instance t^1 driven by the control $u^{\alpha_k, k}(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$:

$$x^{\alpha}(t^1) = F(t^1)F^{-1}(\tau)z + \int_{\tau}^{t^1} F(t^1)F^{-1}(s)bu^{\alpha_k, k}(s \mid \tau, z, 0)ds, \quad t \in T^0(\tau),$$

by I_{a} denote the set of active constraints imposed on the control $u^{\alpha_k, k}(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$ in problem (5.3):

$$I_{\text{a}} = \{i \in \{0, 1, \dots, M(k)\} : h_{(i)}^{1\prime} x^{\alpha_k}(t^1) = \bar{\beta}_i^{\alpha_k}\},$$

and let ν_i^0 $(i \in I_{\text{a}})$ be the Lagrange multipliers found by solving problem (5.3) (see [14]) ($\nu_i^0 = 0$ for $i \in \{0, 1, \dots, M(k)\} \backslash I_{\text{a}}\}$).

Case 1. If $|I_{\text{a}}| > 1$, we set $M(k+1) = M(k) + 1$ and add the vector

$$h_{(M(k+1))}^1 = -\frac{\Sigma_{i \in I_{\text{a}}} \nu_i^0 h_{(i)}^1}{\left\| \Sigma_{i \in I_{\text{a}}} \nu_i^0 h_{(i)}^1 \right\|_2} \tag{5.4}$$

to the system of normal vectors determining the polyhedron $X_{\alpha}^{1, k}$.

Case 2. If $I_{\text{a}} = \{i_0\}$ $(i_0 > m)$, then find the point $z_{i_0}^0$ of contact of the closure of $X_{\alpha_k}^1$ and the polyhedron $X_{\alpha_k}^{1, k}$. At this point, the minimum of the objective function of problem (5.2) is achieved for $\alpha = \alpha_k$, $i = i_0$. Construct the vector $s = z_{i_0}^0 - x^{\alpha_k}(t^1)$ and find the vectors

$$h_{(M(k)+i)}^1 = \left(h_{(i_0)}^1 + \frac{i}{p}s\right)\left(\left\|h_{(i_0)}^1 + \frac{i}{p}s\right\|_2\right)^{-1}, \quad i = \overline{1, p}$$

(here, p is the parameter of the method, which was set to two in the numerical experiments). Set $M(k+1) = M(k) + p$.

Case 3. If $I_{\text{a}} = \{i_0\}$ $(i_0 < m)$, then we construct the vectors

$$h_{(M(k)+i)}^1 = \frac{h_{(i_0)}^1 + \delta h_{(l)}^1}{\left\|h_{(i_0)}^1 + \delta h_{(l)}^1\right\|_2}, \quad i = \overline{1, m}, \quad l = \begin{cases} i-1, & i < i_0, \\ i, & i > i_0, \end{cases} \tag{5.5}$$

where $\delta > 0$ is a small number. Set $M(k+1) = M(k) + m$.

Set $k := k + 1$, and execute the first procedure of the algorithm beginning with the value $\alpha = \alpha_{k-1}$ found earlier. In cases 2 and 3, we will use the optimal associated elements of the i_0th problem (5.2) constructed for α_{k-1} as the initial approximation of the associated elements of the optimal control for the next set of problems (5.2). In case 1, the simplest way of constructing the associated elements is to find an $i_0 \in I_{\text{a}}$ such that $\left|\nu_{i_0}^0\right| = \max_{i \in I_{\text{a}}} \left|\nu_i^0\right|$ and use the associated elements of the i_0th problem (5.2).

If it turns out that the inequalities

$$\alpha_{k-1} - \alpha_k \le \varepsilon, \quad \left\|x^{\alpha_k}(t^1) - x^{\alpha_{k-1}}(t^1)\right\| \le \varepsilon, \tag{5.6}$$

where $\varepsilon \ge 0$ is the given accuracy, hold for the resulting α_k, then we complete the solution and set $\alpha^0 = \alpha_k$ and $u^{\alpha^0}(t \mid \tau, z, 0) = u^{\alpha_k, k}(t \mid \tau, z, 0)$ $(t \in T^0(\tau))$.

If $\alpha_{k-1} - \alpha_k > \varepsilon$, we execute the second procedure.

Remark 1. In the course of iterations, it is reasonable to eliminate the auxiliary vectors $h_{(i)}^1$ $(i > m)$ to reduce the number of passive constraints in problem (5.3). To this end, we choose a number $M_0 > m$ and remove the vectors with the indices in the set $\{m + 1, \dots, M(k)\} \backslash I_{\text{a}}$ from the system $h_{(i)}^1$ $(i = \overline{0, M(k)})$ when

$M(k) > M_0$. This procedure reduces the number of problems (5.2) and the number of constraints in problem (5.3).

5.2. Implementation of the Optimal Closable Feedback

Denote by X_τ^α the set of all $z \in \mathbb{R}^n$ for which there exists optimal open-loop controls in problem (2.5) with the closing instance t^1.

The function

$$u^0(\tau, z) = \begin{cases} u^{\alpha^0}(\tau \mid \tau, z, 0), & z \in X_\tau^\alpha, \quad \tau \in T_h^0(t_*), \\ u^0(\tau \mid \tau, z), & z \in X_\tau, \quad \tau \in T_h^1, \end{cases} \tag{5.7}$$

will be called the optimal closable feedback with the closing instance t^1.

As in Section 3, the function $u^*(\tau) = u^*(\tau, x^*(\tau))$ $(\tau \in T_h)$ will be called the realization of the optimal feedback (5.7) in the particular control process.

Up to $\tau = t_*$, the optimal controller constructs the control $u^{\alpha^0}(t \mid t_*, x_0^*, 0)$ $(t \in T^0(t_*))$ using the algorithm described above and feeds the control $u^*(t_*) = u^{\alpha^0}(t_* \mid t_*, x_0^*, 0)$ at the input of system (2.3) on the time interval $[t_*, t_* + h[$. At an arbitrary instance $\tau \in T_h^0$, the controller constructs $u^{\alpha^0}(t \mid \tau, x^*(\tau), 0)$ $(t \in T^0(\tau))$ using the same algorithm. When doing so, the optimal value α^0 obtained at the instance $\tau - h$ is used as the initial value, and the initial approximations of the associated elements of the optimal controls in problems (5.2) and (5.3) are set to the optimal associated elements constructed at $\tau - h$. Having constructed the control $u^{\alpha^0}(t \mid \tau, x^*(\tau), 0)$ $(t \in T^0(\tau))$, the controller feeds the signal $u^*(\tau) = u^{\alpha^0}(\tau \mid \tau, x^*(\tau), 0)$ at the input of system (2.3) on the interval $[\tau, \tau + h[$.

Beginning with the instance t^1, the controller uses the solution to problem (3.5) for the state $(\tau, x^*(\tau))$ to produce the control $u^*(\tau)$ $(\tau \in T_h^1)$. In order to construct the control $u^0(t \mid t^1, x^*(t^1))$ $(t \in T^1)$ in real time, simultaneously with the calculation of $u^*(\tau)$ $(\tau \in T_h^0)$, approximations of the associated elements of this control are calculated. For this purpose, for each realized state $(\tau, x^*(\tau))$ $(\tau \in T_h^0)$, we find the state

$$x(t^1 \mid \tau, x^*(\tau)) = F(t^1)F^{-1}(\tau)x^*(\tau) + \int_\tau^{t^1} F(t^1)F^{-1}(s)b(s)u^{\alpha^0}(s \mid \tau, x^*(\tau), 0)ds$$

and construct the optimal control $u^0(t \mid t^1, x(t^1 \mid \tau, x^*(\tau)))$ $(t \in T^1)$ in problem (3.5) for the state $(t^1, x(t^1 \mid \tau, x^*(\tau)))$. Therefore, by the instance $\tau = t^1$, we have the optimal associated elements in problem (5.3) for the state $(t^1, x(t^1 \mid t^1 - h, x^*(t^1 - h)))$. The difference in the states $x(t^1 \mid t^1 - h, x^*(t^1 - h))$ and $x^*(t)$ depends only on a bounded perturbation on the time interval h; therefore, using the available information, the controller is able to produce the control $u^0(t \mid t^1, x^*(t^1))$ $(t \in T^1)$ in time not exceeding h time units.

6. EXAMPLE 1

We illustrate the methods of guaranteed optimization described above by solving the problem of maximum acceleration of an oscillation system. The mathematical model of this system has the form

$$\begin{gathered} x_2(t^*) \longrightarrow \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u + w, \\ x(0) = x_0^*, \quad x_1(t^*) \in X^* = \{x_1 \in \mathbb{R} : x_{1*} \le x_1 \le x_1^*\}, \\ |u(t)| \le 1, \quad w(t) \in W = \{w \in \mathbb{R} : |w| \le w^*\}, \quad t \in T. \end{gathered} \tag{6.1}$$

The parameters are as follows: $x_0^* = (0, 1)$, $t^* = 12$, $x_{1*} = 2$, $x_1^* = 7$, and $w^* = 0.5$.

First, we illustrate the construction of the optimal open-loop control $u^{\alpha^0}(t \mid 0, x_0^*)$ $(t \in T^0)$. The closing instance is set to $t^1 = 8$. Upon executing the first procedure of the algorithm, we obtain $\alpha_1 = 2.751587$ and the control

$$u^{\alpha_1, 1}(t \mid 0, x_0^*, 0) = \begin{cases} 1, \quad t \in [0, 0.16[\cup [3.36, 6.48[, \\ -0.074369, \quad t \in [0.16, 0.24[, \\ -1, \quad t \in [0.24, 3.36[\cup [6.48, 8[. \end{cases} \tag{6.2}$$

It is seen in Fig. 1 that the closing set $X^1_{\alpha_1}$ is bounded by the curve *AKLMND*. The approximating set $X^{1,1}_{\alpha_1}$ is given by the polygonal line *ABCD*. The line *A'B'C'D'* determines the restricted set $\bar{X}^{1,1}_{\alpha_1}$, where the trajectory of the determinate system (5.1) must arrive by the time t^1. Indeed (see Fig. 1), under the control $u^{\alpha_1, 1}(t \mid 0, x_0^*, 0)$ $(t \in T^0)$, the trajectory of system (5.1) arrives from the state x_0^* to the point $x^{\alpha_1}(t^1) = C'$. The set of states $X^1_{\alpha_1}(0, x_0^*)$ that system (6.1) can be in at the instance t^1 under the control $u^{\alpha_1, 1}(t \mid 0, x_0^*, 0)$ $(t \in T^0)$ and any perturbations $w(t) \in W$ $(t \in T^0)$ is also shown in Fig. 1. It is seen that $X^1_{\alpha_1}(0, x_0^*) \subset X^1_{\alpha_1}$; i.e., control (6.2) is the optimal open-loop control in problem (2.5). To demonstrate it, we execute the second procedure of the algorithm. The set of active constraints in problem (5.3) is $I_a = \{0, 1\}$, and the optimal Lagrange multipliers are $\nu_0^0 = -0.145131$ and $\nu_1^0 = -0.139710$. Find the new normal vector $h^1_{(3)} = (0.053955, -0.998543)$ and form the set $X^{1,1}_{\alpha_1}$. In Fig. 1, this set is shown by the polygonal line *ABPQD*. The restriction of $X^{1,1}_{\alpha_1}$ in the direction $h^1_{(3)}$ yields the line *P'Q'*. Thus, the point *C'* again belongs to the restricted set $\bar{X}^{1,1}_{\alpha_1}$. Therefore, solving problems (5.2) and (5.3), we obtain $\alpha_2 = \alpha_1$ and $x^{\alpha_2}(t^1) = x^{\alpha_1}(t^1)$; i.e., $\alpha^0 = \alpha_1$ is the optimal guaranteed value of the objective function, and control (6.2) is the optimal open-loop control.

Now, set $t^1 = 4$. Executing the first procedure of the algorithm, we obtain $\alpha_1 = 3.865088$ and

$$u^{\alpha_1, 1}(t \mid 0, x_0^*, 0) = \begin{cases} 1, \quad t \in [0, 0.4[\cup [3.56, 4[, \\ 0.870709, \quad t \in [3.52, 3.56[, \\ -1, \quad t \in [0.4, 3.52[. \end{cases} \tag{6.3}$$

Figure 2 shows the sets $X^1_{\alpha_1}$, $X^{1,1}_{\alpha_1}$, $\bar{X}^{1,1}_{\alpha_1}$, $X^1_{\alpha_1}(0, x_0^*)$ and the trajectory of the determinate system (5.1) corresponding to control (6.3) (the notation is the same as in Fig. 1). In contrast to the preceding case, we

Table 1

k	α_k	$x^{\alpha_k}(t^1)$
1	3.865088	–2.672962, 0.393520
2	2.772491	–2.670667, 0.398154
3	2.707208	–2.348972, 0.926592
4	2.598492	–2.336955, 0.943420
5	2.593512	–2.256832, 1.049475
6	2.583112	–2.273043, 1.028590
7	2.581439	–2.200702, 1.119911
8	2.579865	–2.196962, 1.124546
9	2.579865	–2.196957, 1.124554

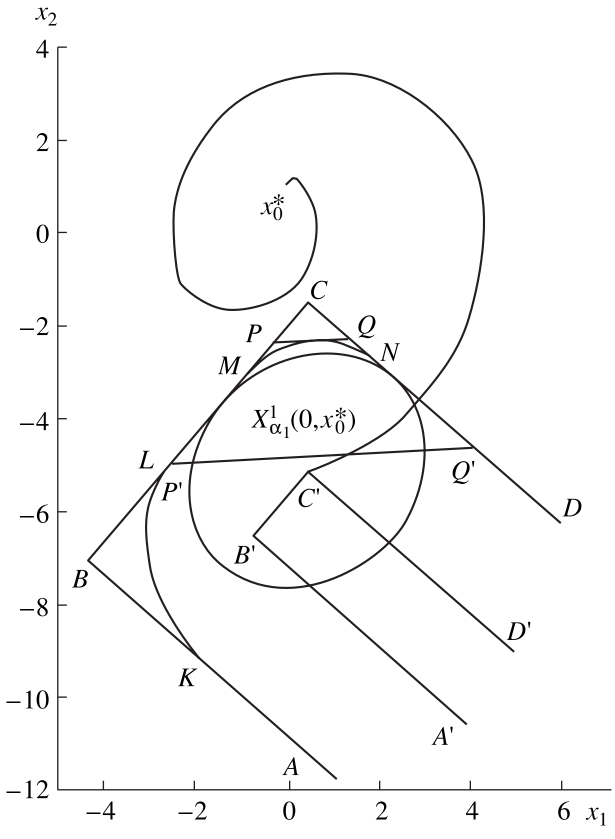

Fig. 1.

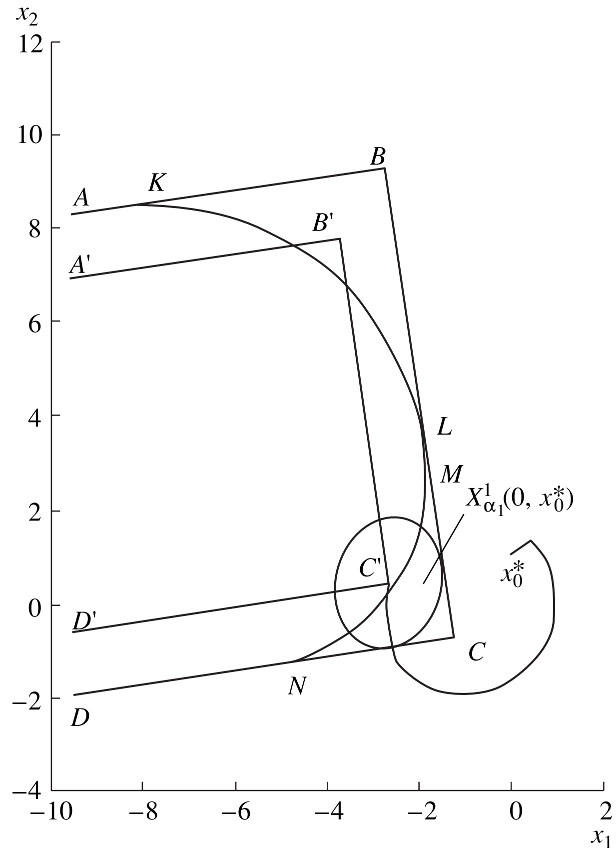

Fig. 2.

now have $X^1_{\alpha_1}(0,\ x_0^*) \not\subset X^1_{\alpha_1}$. For this reason, nine iterations of the algorithm were required to solve the problem, and the following optimal open-loop control was produced:

$$u^{\alpha^0}(t \mid 0, x_0^*, 0) = \begin{cases} 1, \quad t \in [0, 0.16[\cup [3.32, 4[, \\ -0.724368, \quad t \in [0.16, 0.20[, \\ -1, \quad t \in [0.2, 3.32[. \end{cases} \tag{6.4}$$

Table 1 presents the values of α_k and $x^{\alpha_k}(t^1)$ $(k = \overline{1, 9})$ for each iteration step.

Figure 3 shows the sets $X^1_{\alpha_9}$, $X^{1,9}_{\alpha_9}$, $\bar{X}^{1,9}_{\alpha_9}$, $X^1_{\alpha_9}(0,\ x_0^*)$ and the trajectory of the determinate system (5.1) corresponding to control (6.4).

Proceed to solving the problem of guaranteed closed-loop optimization of system (6.1). The set $\bar{X}^*(0, x_0^*)$ is empty; therefore, the optimal unclosable feedback is impossible to construct. Consider the behavior of the system closed by the two-phase unclosable and closable feedback.

Assume that the system is affected by an unknown (to the controller) perturbation $w^*(t) = 0.5\sin(2t)$ $(t \in T)$. The realization of the two-phase unclosable feedback is calculated by solving the first-phase problem (4.2) up to the time $\bar{t} = 4.16$: $u^*(\tau) = u^0_\gamma(\tau \mid \tau, x^*(\tau))$ $(\tau \in T^0_h(0))$. In Fig. 4, curve *1* depicts the realized values $\gamma^0(\tau, x^*(\tau))$ $(\tau \in [0, \bar{t}\,[)$. At the instance $\bar{t}$, the second control phase starts, and the solution of problem (3.5) is used to calculate $u^*(\tau) = u^0(\tau \mid \tau, x^*(\tau))$. Curve *2* depicts the guaranteed values of the objective function (3.3) along the optimal controls $u^0(t \mid \tau, x^*(\tau))$ $(t \in T(\tau), \tau \in [\bar{t}, t^*[)$. This control strategy made it possible to reach the value of the objective function $h'_{(0)}x^*(t^*) = 5.462913$ by the end of the process. The control fed at the input of system (6.1) was

$$u^*(\tau) = \begin{cases} -1, \quad \tau \in [0, 2.56[\cup [7.2, 10.4[, \\ -0.55866, \quad \tau \in [7.12, 7.2[, \\ 1, \quad \tau \in [2.56, 7.12[\cup [10.4, 12[. \end{cases}$$

The effectiveness of the closable feedback is illustrated in Fig. 4 by curve *3*, which shows the optimal values of $\alpha^0(\tau)$ along the realized trajectory $x^*(\tau)$ $(\tau \in T)$. The closing instance was set to $t^1 = 8$, and the following realization was constructed:

$$u^*(\tau) = \begin{cases} 1, \quad \tau \in [0, 0.16[\cup [3.52, 6.88[\cup [10.4, 12[, \\ 0.651871, \quad \tau \in [0.16, 0.24[, \\ -1, \quad \tau \in [0.24, 3.44[\cup [6, 88, 10.4[, \\ 0.515574, \quad \tau \in [3.44, 3.52[. \end{cases}$$

By the time t^*, the objective function reached the value $\alpha^0(t^*) = 7.249922$.

Note that the optimal open-loop control for the determinate system (6.1), with the known perturbation $w^*(t)$ $(t \in T)$, yields the value of the objective function equal to 7.793629.

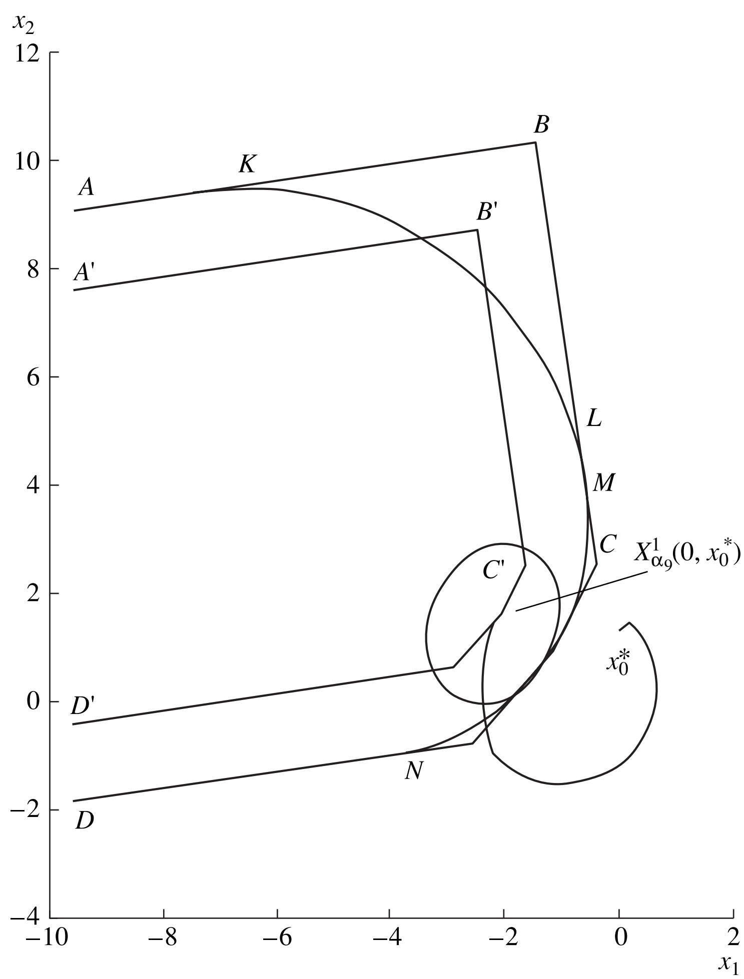

Fig. 3.

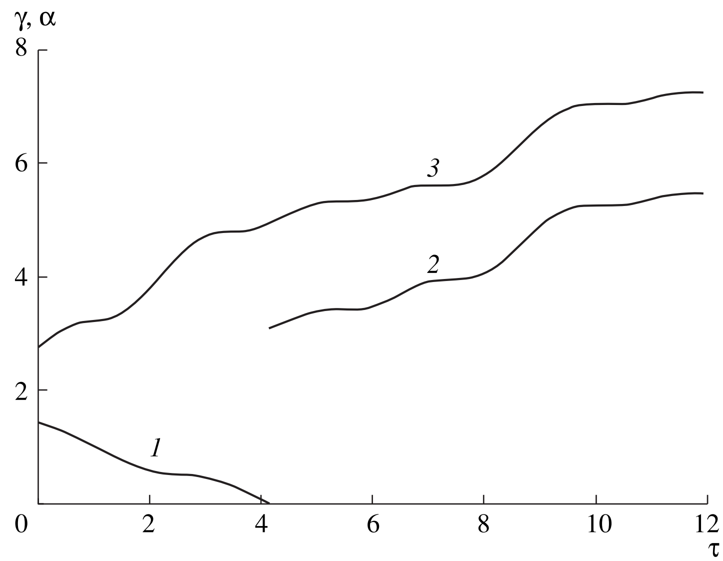

Fig. 4.

7. OPTIMAL MULTIPLY CLOSABLE FEEDBACK

Consider the problem of synthesizing the optimal feedback with several closing instances

$$\{t^i \in T_h, i = \overline{1, p}\}, \quad t_* < t^1 < \ldots < t^p < t^* = t^{p+1}. \tag{7.1}$$

In other words, when constructing the feedback, we will use the fact that the system will be closed not only at the current instance $\tau \in T_h$, but also at all instances (7.1). We also assume that the available information can be processed in real time. We assign the following closing sets to the closing instances:

$$X^1, \ldots, X^p. \tag{7.2}$$

Sets (7.2) are determined recurrently. Define $X^{p+1} = X^*$. The set X^i contains the states $x \in \mathbb{R}^n$ that can be surely taken to the set X^{i+1} on the time interval $T^i = [t^i, t^{i+1}[$.

The optimal feedback constructed with the use of p closing instances is called the optimal multiply closable feedback. When $p = N$, it is called the optimal closed feedback. Closed feedbacks are used in dynamic programming.

We explain the changes that should be made in the constructs discussed in Section 5 for the case of two closing instances ($p = 2$). The generalization for any number of closing instances is similar, but is not presented here due to its awkwardness.

Take two closing instances $t^1, t^2 \in T_h$ ($\tau < t^1 < t^2 < t^* - h$), and define the closing α-sets X_α^2 and X_α^1 following Section 5; we also define α-admissible controls $u^\alpha(t \mid t^1, x, 1)$ ($t \in T^1, x \in X_\alpha^1$) and $u^\alpha(t \mid t^2, x, 2)$ ($t \in T^2, x \in X_\alpha^2$).

For convenience, we set $t^0 = \tau$, $T^0 = T^0(\tau)$, and $h_{(i)}^3 = h_{(i)}$ ($i = \overline{0, m}$).

The maximum value of α^0 for which there exist the sets $X_{\alpha^0}^1$, $X_{\alpha^0}^2$ and an α^0-admissible control $u^{\alpha^0}(t \mid \tau, z, 0)$ ($t \in T^0(\tau)$) is the optimal guaranteed value of the objective function in problem (2.5) for the closing instances t^1 and t^2.

By $X_\alpha^2(x)$, we denote the set of states that system (2.5) can be in when it starts from the state (t^1, x) and is driven by the control $u^\alpha(t \mid t^1, x, 1)$ ($t \in T^1$) under the influence of arbitrary realizations of the perturbation $w(t) \in W$ ($t \in T^1$).

The optimal open-loop control in problem (2.5) with the closing instances t^1 and t^2 is the α^0-admissible control $u^{\alpha^0}(t \mid \tau, z, 0)$ ($t \in T^0(\tau)$) and the families of α^0-admissible controls $u^{\alpha^0}(t \mid t^1, x, 1)$ ($t \in T^1$) and $u^{\alpha^0}(t \mid t^2, y, 2)$ ($t \in T^2$, $y \in X^2_{\alpha_0}(x)$, $x \in X^1_{\alpha_0}(\tau, z)$).

Let us describe the changes to Algorithm 1 designed to calculate α^0 and construct the control $u^{\alpha^0}(t \mid \tau, z, 0)$ ($t \in T^0(\tau)$) for the case of two closing instances.

Set $k = 1$ and $M(k) = m$.

In order to construct the approximating polyhedra $X^{2,k}_\alpha$ and $X^{1,k}_\alpha$, solve problems (5.2) first on the interval T^2 and then on T^1:

$$\begin{gathered}\beta_i^{l\alpha} = \min_{y,u} h_{(i)}^{l\prime} y, \quad \dot{x} = A(t)x + b(t)u, \quad x(t^l) = y, \\ x(t^{l+1}) \in \bar{X}_\alpha^{l+1,k}, \quad |u(t)| \le 1, \quad t \in T^l, \quad i = \overline{0, M(k)}, \quad l = 2, 1,\end{gathered} \tag{7.3}$$

where

$$h_{(i)}^{l\prime} = h_{(i)}^{l+1\prime} F(t^{l+1} - t^l) / \| F'(t^{l+1} - t^l) h_{(i)}^{(l+1)} \|_2, \quad i = \overline{0, m}, \quad l = 2, \quad i = \overline{0, M(k)}, \quad l = 1. \tag{7.4}$$

The set $\bar{X}^{3,k}_\alpha$ is taken to be equal to $\bar{X}^*_\alpha$. The sets $X^{2,k}_\alpha$ and $\bar{X}^{2,k}_\alpha$ are constructed upon solving problems (7.3) for $l = 2$:

$$X_\alpha^{l,k} = \{x \in \mathbb{R}^n : h_{(i)}^{l\prime} x \ge \beta_i^{l\alpha}, i = \overline{0, M(k)}\}, \quad \bar{X}_\alpha^{l,k} = \{x \in \mathbb{R}^n : h_{(i)}^{l\prime} x \ge \bar{\beta}_i^{l\alpha}, i = \overline{0, M(k)}\}, \tag{7.5}$$

where

$$\bar{\beta}_i^{l\alpha} = \beta_i^{l\alpha} - \min_{w(t) \in W, t \in T^{l-1}} \int_{t^{l-1}}^{t^l} h_{(i)}^{l\prime} F(t^l - s) d(s) w(s) ds.$$

Then, upon solving problems (7.3) for $l = 1$, we find, according to (7.5), the sets $X^{1,k}_\alpha$ and $\bar{X}^{1,k}_\alpha$.

Problem (5.3) remains the same, and the algorithm for constructing a k-suboptimal control in problem (2.5) is as follows.

Algorithm 2

Step 1. Set $\bar{\rho} = 0$.

Step 2. For $l = 2, 1$, perform the following actions:

2.1. Solve problem (5.2) $M(k) + 1$ times.

2.2. Form the set $\bar{X}^{l,k}_\alpha$.

2.3. If $\bar{X}^{l,k}_\alpha = \varnothing$, then check whether it is possible to make $\bar{X}^{l,k}_\alpha \ne \varnothing$ by decreasing α. If this is possible, we set $\alpha_{\max} = \alpha$, decrease α, and go to Step 2.1. Otherwise, there is no guaranteed control for the closing instances t^1 and t^2.

Step 3. Solve problem (5.3) to find ρ_α.

Step 4. If $\alpha = \alpha_{\max}$ and $\rho_\alpha \le 1$ or $1 - \varepsilon \le \rho_\alpha \le 1$ (ε is the given accuracy), then we complete the solution and set $\alpha_k = \alpha$ and $u^{\alpha_k, k}(t \mid \tau, z, 0) = u^0_{\alpha_k}(t)$ ($t \in T^0(\tau)$).

If $\rho_\alpha = \bar{\rho} > 1$, then there are no admissible controls in problem (2.5).

If $\rho_\alpha > 1$, we decrease α. If $\rho_\alpha < 1 - \varepsilon$, we increase α, set $\bar{\rho} = \rho_\alpha$, and go to Step 2.

The increment of α is decreased as ρ_α approaches unity.

The main changes to the algorithm due to the introduction of the second closing instance concern the second procedure. In order to refine the structure of the sets $X^{1,k}_\alpha$ and $X^{2,k}_\alpha$, we first follow certain rules to complete the system of normal vectors $h^2_{(i)}$ $(i = \overline{0, M(k)})$ that determine the set $X^{2,k}_\alpha$, and then use formula (7.4) to calculate the corresponding vectors $h^1_{(i)}$ $(i = \overline{M(k)+1, M(k+1)})$.

As in Section 5, the first procedure yields the point $x^\alpha(t^1)$, the set I_a of active constraints of problem (5.3), and the optimal Lagrange multipliers ν^0_i $(i \in I_a)$ in problem (5.3).

Using rules (5.4) and (5.5) (substituting h^2 for h^1 and using rule (5.5) each time when $|I_a| = 1$) we find the new normal vectors $h^2_{(i)}$ $(i = \overline{M(k)+1, M(k+1)})$ on the basis of I_a and the Lagrange multipliers ν^0_i $(i \in I_a)$.

Furthermore, we find the spread of states of system (2.5) at the time t^1 with respect to the point $x^\alpha(t^1)$ in the directions $h^1_{(i)}$ $(i = I_a)$ under the influence of the perturbations $w(t) \in W$ $(t \in T^0(\tau))$. To this end, we solve $|I_a|$ linear programming problems for the state x:

$$h^{1\prime}_{(i)}x \longrightarrow \max_x, \quad \int_\tau^{t^1} F(t^1 - s)d(s)w(s)ds - x = -x^\alpha(t^1), \quad w(t) \in W, \quad t \in T^0(\tau). \tag{7.6}$$

The optimal solution for the ith problem (7.6) is denoted by x^i_w.

For the points x^i_w $(i \in I_a)$, we apply the algorithm described above beginning with α_k (which is already calculated) and using the same notation for t^2 as was used for t^1. The first procedure yields $\alpha(x^i_w)$. In contrast to subsection 5.1, if the objective function of problem (5.3) satisfies the inequality $\rho_\alpha < 1$ for the initial $\alpha = \alpha_k$, we do not increase α, but complete the first procedure by setting $\alpha(x^i_w) = \alpha_k$. When passing to the next point x^{i+1}_w, we refine $\alpha_k : \alpha_k := \min\{\alpha_k, \alpha(x^i_w)\}$.

For every point x^i_w $(i \in I_a)$, we execute the second procedure using the results obtained by the first procedure; namely, we complete the system of vectors $h^2_{(i)}$ $(i = \overline{0, M(k+1)})$. Thus, we obtain a complete set of normal vectors $h^2_{(i)}$ $(i = \overline{0, M(k+1)})$, and find the corresponding vectors $h^1_{(i)}$ $(i = \overline{M(k)+1, M(k+1)})$ by formula (7.4). Then, set $k := k + 1$ and go to the next iteration step of the basic algorithm, starting the first procedure from the refined value $\alpha = \alpha_{k-1}$.

The solution is completed when condition (5.6) is satisfied.

To determine the optimal closable feedback, we denote by X^α_τ the set of all $z \in \mathbb{R}^n$ for which problem (2.5) has optimal open-loop controls with the closing instances t^1 and t^2 in the case $\tau \in T^0_h(t_*)$ and with the closing instance t^2 in the case $\tau \in T^1_h = \{t^1, t^1 + h, \ldots, t^2 - h\}$.

The function

$$u^0(\tau, z) = \begin{cases} u^{\alpha^0}(\tau \mid \tau, z, 0), \quad z \in X^\alpha_\tau, \quad \tau \in T^0_h(t_*), \\ u^{\alpha^0}(\tau \mid \tau, z, 1), \quad z \in X^\alpha_\tau, \quad \tau \in T^1_h, \\ u^0(\tau \mid \tau, z), \quad z \in X_\tau, \quad \tau \in T^2_h = \{t^2, t^2 + h, \ldots, t^* - h\}, \end{cases}$$

is called the optimal closable feedback with the closing instances t^1 and t^2.

The realization of the optimal feedback $u^*(\tau) = u^*(\tau \mid \tau, x^*(\tau))$ $(\tau \in T_h)$ is constructed by the scheme described in subsection 5.2. Simultaneously with the computation of $u^*(\tau)$ $(\tau \in T^0_h(t_*))$, some preliminary auxiliary computations are performed. More precisely, the state $x(t^1 \mid \tau, x^*(\tau))$ is evaluated, the optimal open-loop control $u^{\alpha^0}(t \mid t^1, x(t^1 \mid \tau, x^*(\tau)), 1)$ $(t \in T_h^1)$ for this state is calculated, the point

$$x(t^2 \mid t^1, x(t^1 \mid \tau, x^*(\tau))) = F(t^2)F^{-1}(t^1)x(t^1 | \tau, x^*(\tau)) + \int_{t^1}^{t^2} F(t^2)F^{-1}(s)b(s)u^{\alpha^0}(s \mid t^1, x(t^1 \mid \tau, x^*(\tau)), 1)ds$$

is found, and the optimal open-loop control $u^0(t \mid t^1, x(t^2 \mid t^1, x(t^1 \mid \tau, x^*(\tau))))$ $(t \in T^2)$ is constructed. This auxiliary information makes it possible to switch to another way of control at the time t^1 and t^2 in real time.

Remark 2. The control system (2.1) with constraints imposed on the control and perturbation is called effective if, for any sufficiently small $h > 0$ and the closing instances $t_* = t^0 < \dots t^p < t^*$, the maximal radius (r_k) of the ball that can be placed in the closing set X^k increases with decreasing index k. For effective control systems, for any terminal set X^* and $\varepsilon > 0$, one can construct an optimal closable feedback such that the system closed by this feedback is brought from the state $x(t_*) \in X^0$ (where X^0 is the closing set at the time t^0 constructed from the set X^1 by the rule specified above) into the ε-neighborhood of X^* for an arbitrary realization of the perturbation $w(t) \in W$ $(t \in T)$. This is the main advantage of closable feedbacks over unclosable feedbacks. However, unclosable feedbacks have a very useful property, which is not inherent in closable feedbacks: if the processor of the optimal controller fails, the open-loop control that was used to construct the realization of the optimal unclosable feedback at this moment brings the system to the terminal set independently of the future perturbations.

8. EXAMPLE 2

Again, consider problem (6.1). Assume that the initial state is $x_0^* = (-1, -1)$. Construct the optimal open-loop control $u^{\alpha^0}(t \mid 0, x_0^*, 0)$ $(t \in T^0(0))$ for two closing instances $t^1 = 6$ and $t^2 = 9$. Figure 5 shows closing α_1-sets $X_{\alpha_1}^1$ and $X_{\alpha_1}^2$ bounded by the polygonal lines *AKLMND* that approximate the sets $X_{\alpha_1}^{1,1}$ and $X_{\alpha_1}^{2,1}$ (the curves *ABCD*), and the restricted sets $\bar{X}_{\alpha_1}^{1,1}$ and $\bar{X}_{\alpha_1}^{2,1}$ (the polygonal lines *A'B'C'D'*). On the basis of the

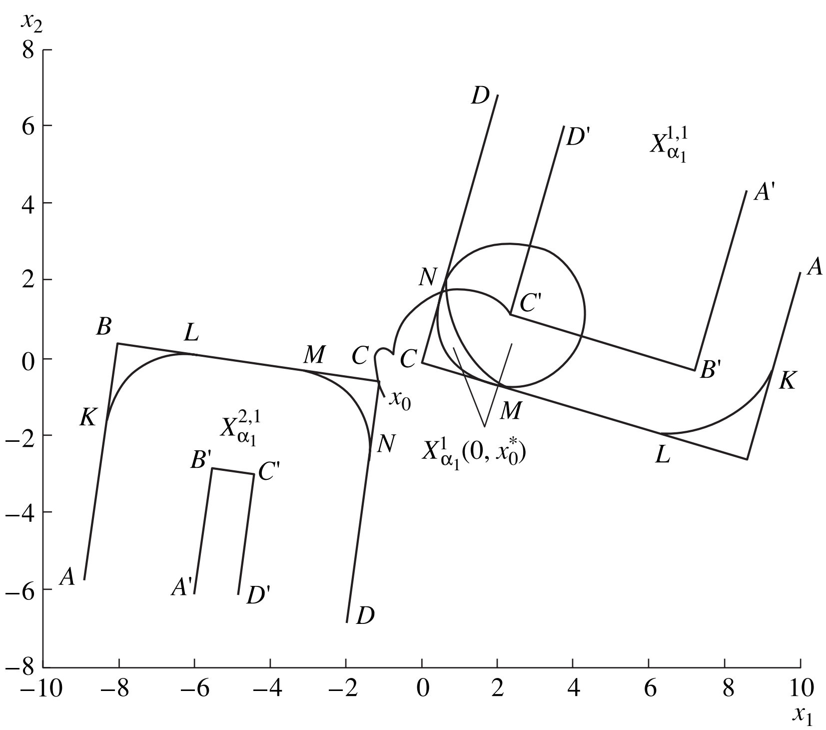

Fig. 5.

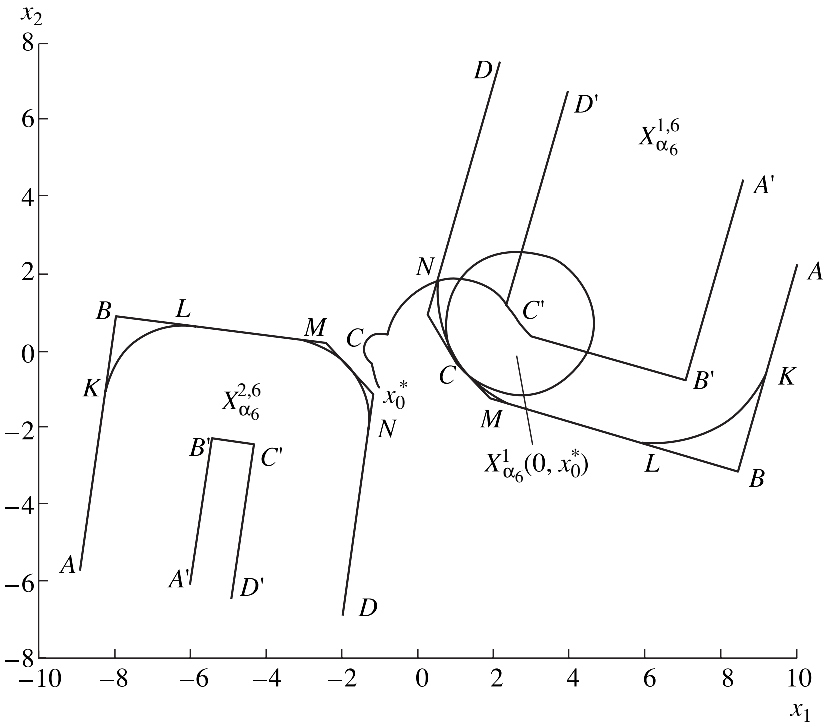

Fig. 6.

set $\overline{X}^{1,1}_{\alpha_1}$, the control

$$u^{\alpha_1,1}(t \mid 0, x_0^*, 0) = \begin{cases} 1, & t \in [0, 0.42[\cup [3.6, 6[, \\ -1, & t \in [0.42, 3.54[, \\ -0.778181, & t \in [3.54, 3.6[\end{cases}$$

was constructed. Under this control, the trajectory of the determinate system (5.1) arrives at the point C' at the time t^1. Figure 5 also shows the set $X^1_{\alpha_1}(0, x_0^*)$ of possible states of the perturbed system (6.1) at the time t^1. It is seen that $X^1_{\alpha_1}(0, x_0^*) \not\subset X^1_{\alpha_1}$.

The optimal open-loop control $u^{\alpha^0}(t \mid 0, x_0^*, 0)$ $(t \in [0, 6[)$ was constructed in six iteration steps; it has the form

$$u^{\alpha^0}(t \mid 0, x_0^*, 0) = \begin{cases} 1, & t \in [0, 0.3[\cup [3.48, 6[, \\ -1, & t \in [0.3, 3.42[, \\ -0.388118, & t \in [3.42, 3.48[. \end{cases} \tag{8.1}$$

Table 2

k	α_k	$x^{\alpha_k}(t^1)$
1	1.720427	2.368126, 1.115149
2	1.221050	2.368125, 1.115145
3	1.184795	2.585360, 0.845156
4	1.175652	2.699952, 0.679233
5	1.175554	2.700842, 0.677907
6	1.175554	2.700842, 0.677907

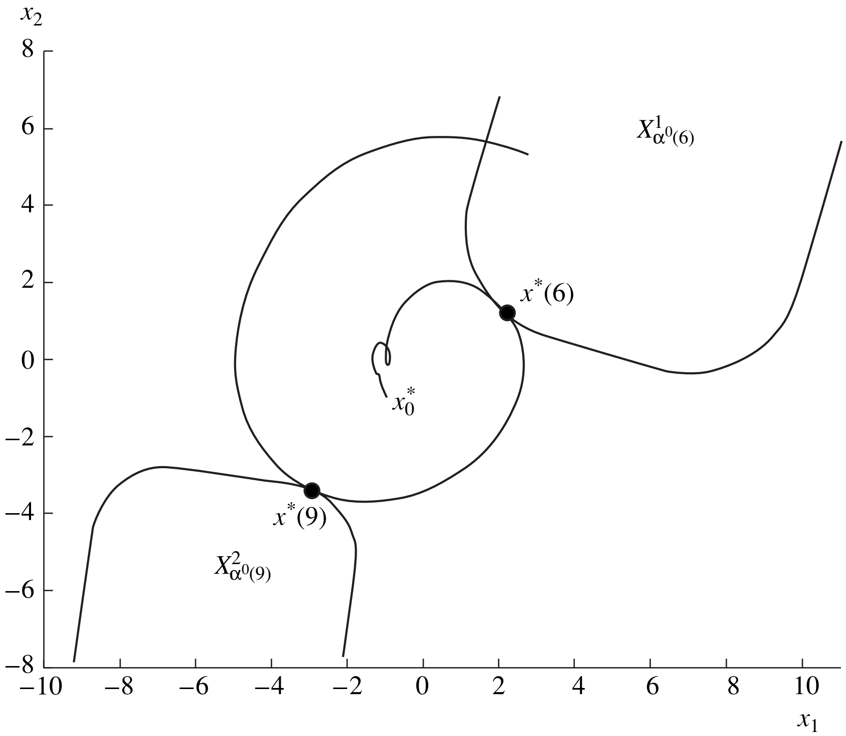

Fig. 7.

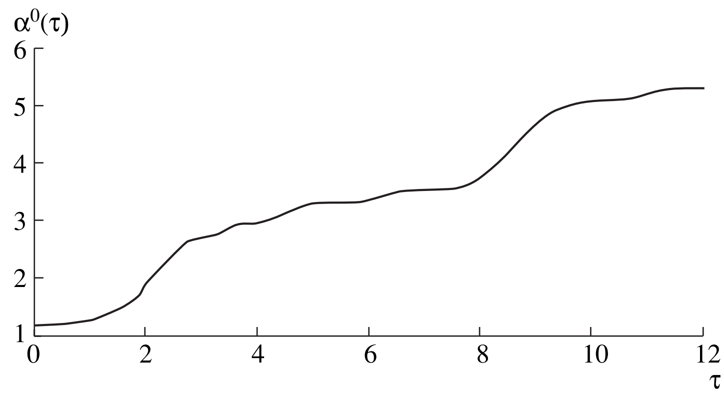

Fig. 8.

Table 2 presents the values of α_k and $x^{\alpha_k}(t^1)$ $(k = \overline{1, 6})$ for each iteration step.

The sets $X^i_{\alpha_6}$, $X^{i,6}_{\alpha_6}$, $\bar{X}^{i,6}_{\alpha_6}$ $(i = 1, 2)$, $X^1_{\alpha_6}(0, x_0^*)$, and the trajectory of the determinate system (5.1) corresponding to control (8.1) are presented in Fig. 6. Thus, the optimal guaranteed value of the objective function is $\alpha^0 = 1.175554$. If the single closing instance $t^1 = 6$ is used instead of two, we have $\alpha^0 = 0.646045$; for the single closing instance $t^1 = 9$, we have $\alpha^0 = 0.733118$.

Now, we construct a realization of the optimal closable feedback. Assume that system (6.1) is affected by an unknown (to the controller) perturbation $w^*(t) = 0.5\sin(2t)$ $(t \in T)$. With two closing instances $t^1 = 6$ and $t^2 = 9$, the controller produced the realization

$$u^*(t) = \begin{cases} 1, & t \in [0, 0.24[\cup [3.84, 6.72[\cup [10.44, 12[, \\ -1, & t \in [0.24, 3.82[\cup [6.84, 10.44[, \\ -0.027471, & t \in [6.72, 6.84[. \end{cases}$$

The corresponding trajectory $x^*(t)$ $(t \in T)$ of the closed system (6.1) is shown in Fig.7. The dots indicate the states $x^*(6)$ and $x^*(9)$. The sets $X^1_{\alpha^0(6)}$ and $X^2_{\alpha^0(9)}$ are also shown.

By the end of the process, the value of the objective function $\alpha^0(t^*) = 5.316653$ was reached. The variation of the optimal guaranteed value $\alpha^0(\tau)$ $(\tau \in T_h)$ along the realized trajectory $x^*(\tau)$ $(\tau \in T)$ is shown in Fig. 8. As described in Section 6, we constructed the optimal open-loop control for the determinate system (6.1) with the given perturbation $w^*(t)$ $(t \in T)$. The optimal value of the objective function was 5.638651.

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