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Indirect Optimal Control of Dynamical Systems

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Abstract—The optimal control problem is investigated for dynamical systems in which controls are not statically (inertialess) transformed signals from a measuring device, but are rather produced by an actuator on the basis of such measurements; the structure of the actuator is described along with the structure of the controlled object. Natural constraints imposed on the input and output signals of the actuator make this problem an optimal control problem with phase constraints. In this respect, indirect control problems differ from direct control problems, which are conventional in the mathematical theory of optimal processes and do not usually involve phase constraints of this type. A special constructive approach to the investigation of indirect control problems is developed. Optimality and suboptimality criteria in the form of the maximum and ε-maximum principles are obtained, fast algorithms for computing open-loop controls are constructed, and algorithms implementing optimal feedback controls are described. Results are illustrated by examples.

INTRODUCTION

In control theory, two types of feedback systems are distinguished—direct control and indirect control systems. In direct control systems, signals from instruments that measure outputs of the controlled object are fed at the input of the controlled object upon inertialess (momentary) transformations. In many cases, such signals are too weak and can be used only for indirect control. More precisely, they are fed at the input of a device that produces controls for an execution unit (servomechanism or actuator) that produces sufficiently powerful controls using an external energy source.

Modern mathematical control theory [1] was developed for direct control systems. Therefore, the development of an optimal indirect control theory is an important problem, as was indicated in [2]. In such a theory, an actuator is assumed to be specified along with the controlled object. A specific feature of this problem is that the formal addition of equations governing the dynamic actuator to the equations governing the controlled object adds phase constraints to the initial problem. Problems with phase constraints are much more complicated than problems without phase constraints.

Nowadays, the theory of necessary optimality conditions for optimal control problems with phase constraints is well developed ([3] is one of the first studies devoted to this problem). However, there are fundamental difficulties that complicate the use of this theory for constructing optimal open-loop controls and synthesis of optimal systems. These difficulties have not yet been overcome despite the efforts directed toward solving this problem during almost fifty years of optimal control theory development.

The aim of this paper is to design constructive actuator-based methods for solving optimal control problems taking into account specific features of actuators. These methods do not use the general mathematical theory of optimal processes; rather, they are based linear programming. It will be shown that specific features of the problem under consideration make it possible to efficiently implement constructive linear programming methods, design fast algorithms for calculating open-loop controls, and generate optimal feedbacks in real time. The results of this paper can be considered as a development of the methods suggested in [4, 5] for direct control systems.

Nowadays, there are powerful software packages for solving linear programming problems. They can be used to obtain optimal solutions for many linear optimal control problems. However, we do not use such software in this paper, since our aim is to produce optimal controls in real time when general-purpose methods are useless and specially designed methods are most efficient.

The paper is organized as follows. In Section 1, classes of feasible controls are described. These controls are produced by a dynamic actuator of a given structure with specified constraints on the input and output signals. The actuator behavior is described by a first-order differential equation, and the input and output signals are subject to geometric constraints. The output signals of the actuator are used as controls fed at the input of the controlled object. A linear terminal optimal control problem is formulated in the class of feasible controls. Open-loop solutions to this problem are obtained in three stages. At the first stage, open-loop solutions to the maximum excitation problem are constructed (Section 2), which differ from the terminal problem by the absence of terminal constraints. The maximum excitation problem illustrates the application of the adaptive linear programming method [6] to optimal control problems. To this end, we use a functional form, which is a special linear programming problem, and thoroughly describe dynamic analogs of the basic elements of the adaptive method (the support (working basis), potential vectors, and estimates). The maximum and ε-maximum principles for optimal and suboptimal controls are formulated as consequences of the optimality and suboptimality linear programming criteria (Subsection 2.2). The support optimality criterion provides a basis for Subsection 2.3. In this subsection, a dual method for calculating the open-loop solution of the maximum excitation problem is developed. It is shown that the proper use of phase constraints makes it possible to construct a method whose complexity is equivalent to the complexity of the similar method developed in [4] for problems without phase constraints. This result is of fundamental importance for this paper, since it allows a practical implementation of the optimal feedback in the terminal optimal control problem with phase constraints. In Subsection 2.4, this method is illustrated by the example of the maximum acceleration of an oscillator. In Subsection 2.5, a refinement procedure is designed and illustrated by way of example. This procedure allows one to construct open-loop solutions of the maximum excitation problem in the class of piecewise continuous controls with a negligible (zero) sampling period. In Section 3, the terminal optimal control problem is investigated. This problem is reduced to the maximum excitation problem using Lagrange multipliers. A procedure for constructing the optimal Lagrange vector is described in detail. Section 4 is devoted to the synthesis of optimal feedback controls. A real-time implementation of the optimal feedback is described. The results of the two last sections are illustrated by examples.

1. ACTUATORS IN DYNAMICAL SYSTEMS: TERMINAL OPTIMAL CONTROL PROBLEM

Let the control system be governed by the equation

$$\dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \tag{1}$$

on the time interval $T = [t_*, t^*]$, where $x = x(t)$ is the n-dimensional state vector of the system at the time t, $u = u(t)$ is the scalar control, $A(t)$ and $b(t)$ ($t \in T$) are a piecewise continuous n-by-n matrix and an n-dimensional vector functions, respectively, and x_0 is the initial state of the system.

The controls $u(t)$ ($t \in T$) are subject to the constraint

$$|u(t)| \le L, \quad t \in T.$$

We assume that they are produced by the actuator

$$\dot{u} = au + v, \quad u(t_*) = u_0, \tag{2}$$

on the basis of bounded control signals $v(t)$

$$|v(t)| \le M, \quad t \in T.$$

The behavior of the actuator and its capabilities of controlling system (1) depend on the parameters a, L, and M. If $|a|L \le M$, then, under the condition $v(t) = \pm M$ for $t \ge t_*$, the solution $u(t)$ ($t \ge t_*$) to Eq. (2) leaves the interval $[-L, L]$ in finite time for any initial value u_0; it can be kept on the boundary using the signal $|v(t)| \equiv |a|L \le M$ ($t \ge t_*$). If $|a|L > M$, then the trajectory $u(t)$ ($t \ge t_*$) cannot be kept on the boundary $u(t) = \pm L$ ($t \ge t_*$) when $a > 0$, which results in the violation of the constraints imposed on the controls. When $a < 0$, the trajectory $u(t)$ ($t \ge t_*$) remains within the interval $[-L, L]$ for any $|u_0| \le L$ and $v(t) = \pm M$ ($t \ge t_*$); therefore, the constraints $|u(t)| \le L$ ($t \in T$) become irrelevant. This analysis shows that only the actuators satisfying the inequality $|a|L \le M$ are of interest for controlling system (1). The set of controls $u(\cdot) = (u(t), t \in T$ produced by such actuators is denoted by U_1.

Remark 1. In this paper, we consider the case $|u(t)| \le L$, $|v(t)| \le M$ for $t \in T$. Two more cases are possible: (1) $|u(t)| \le L$, $|\dot{u}(t)| \le K$ for $t \in T$; (2) $|\dot{u}(t)| \le K$, $|v(t)| \le M$ for $t \in T$. Case (1) is reduced to an optimal control problem in the class of inertial controls. Case (2) leads to mixed constraints and will be studied in another paper.

When calculating optimal control signals $v(t)$ ($t \in T$) for systems that are not very simple (1), digital computers are invariably used; therefore, we assume that the available control signals $v(t)$ ($t \in T$) are discrete functions with the sampling period $h = (t^* - t_*)/N$ (where $N > 0$ is an integer):

$$v(t) \equiv v(t_* + kh), \quad t \in [t_* + kh, t_* + (k+1)h[, \quad k = \overline{0, N-1}.$$

Obviously, the control signal $v(t)$ ($t \in T$) can be uniquely reconstructed from the values $v(t)$ for $t \in T_h = \{t_*, t_* + h, \dots, t^* - h\}$.

The aim of this paper is to develop methods for constructing open-loop and feedback solutions to the problem

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \\ \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \\ \dot{u} = au + v, \quad u(t_*) = u_0, \\ Hx(t^*) = g, \quad |v(t)| \le M, \quad t \in T; \quad |u(s)| \le L, \quad s \in S_h = \{t_* + h, t_* + 2h, \dots, t^*\}, \end{gathered} \tag{3}$$

where $g \in \mathbb{R}^m$, $H \in \mathbb{R}^{m \times n}$, and $\operatorname{rank} H = m < n$.

An available control signal $v(t)$ ($t \in T$) and a control $u(t)$ ($t \in T$) are called feasible in problem (3) if they induce the trajectory $x(t)$ ($t \in T$) of system (1) that arrives at the terminal set

$$x(t^*) \in X^* = \{x \in \mathbb{R}^n : Hx = g\}$$

at the time t^*.

A feasible control signal $v^0(t)$ ($t \in T$) and a control $u^0(t)$ ($t \in T$) are called the optimal open-loop control signal and control, respectively, if they induce the (optimal) trajectory $x^0(t)$ ($t \in T$) that maximizes the objective function

$$c'x^0(t^*) = \max c'x(t^*), \quad u(\cdot) \in U_1.$$

For a given $\varepsilon \ge 0$, a suboptimal (ε-optimal) control signal $v^\varepsilon(t)$ ($t \in T$), control $u^\varepsilon(t)$ ($t \in T$), and the corresponding trajectory $x^\varepsilon(t)$ ($t \in T$) are defined by the inequality $c'x^0(t^*) - c'x^\varepsilon(t) \le \varepsilon$.

The feedback solution to problem (3) is defined in Section 4.

The construction of open-loop and feedback solutions to problem (3) is fairly complicated. First, we find the optimal open-loop control signal and control. The construction of the open-loop solution is performed in three stages. At the first stage, we construct an open-loop solution to the maximum excitation problem that is obtained from problem (3) by ignoring the terminal constraint. At the second stage, we construct a method for calculating an approximate value of the optimal Lagrange vector corresponding to the terminal constraint. Finally, the third stage involves a refinement procedure that constructs the optimal Lagrange vector and the open-loop solution to problem (3). Fast algorithms for the calculation of optimal open-loop controls are then used for the synthesis of optimal systems.

2. MAXIMUM EXCITATION OF DYNAMICAL SYSTEMS

The maximum excitation problem for the dynamical system (1) by means of actuator (2) is defined as the optimal control problem

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \quad u(\cdot) \in U_1, \tag{4}$$

which differs from problem (3) in the absence of terminal constraints.

Interpreting the control u as an additional phase variable x_{n+1}, we write problem (4) in the form of an optimal control problem with a phase constraint:

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \\ \dot{x} = A(t)x + b(t)x_{n+1}, \quad x(t_*) = x_0, \quad \dot{x}_{n+1} = ax_{n+1} + v, \quad x_{n+1}(t_*) = u_0, \\ |x_{n+1}(s)| \le L, \quad s \in S_h, \\ |v(t)| \le M, \quad t \in T. \end{gathered} \tag{5}$$

Using the Cauchy formula [7], we reduce problem (5) to an equivalent functional form

$$\sum_{t \in T_h} c(t)v(t) \longrightarrow \max,$$
$$L_*(s) \le \sum_{t = t_*}^{s-h} d(s-t)v(t) \le L^*(s), \quad s \in S_h, \tag{6}$$
$$|v(t)| \le M, \quad t \in T_h.$$

Here,

$$c(t) = \int_t^{t+h} c'F(t^*)\int_{\vartheta}^{t^*} F^{-1}(\tau)b(\tau)\Phi(\tau)d\tau\Phi^{-1}(\vartheta)d\vartheta, \quad d(t) = \Phi(t)\int_0^h \Phi^{-1}(\vartheta)d\vartheta, \quad t \in T_h,$$

$$L_*(s) = -L - \Phi(s-t_*)u_0, \quad L^*(s) = L - \Phi(s-t_*)u_0, \quad s \in S_h,$$

$F(t)$ $(t \ge t_*)$ is the fundamental matrix of solutions to system (1) ($\dot{F} = A(t)F$, $F(t_*) = E$), and $\Phi(t)$ $(t \ge 0)$ is the fundamental matrix of solutions to Eq. (2) of the dynamic actuator ($\dot{\Phi} = a\Phi$, $\Phi(0) = 1$).

Remark 2. For the first-order actuator considered in this paper, we have the explicit formulas

$$\Phi(t) = e^{at}, \quad d(t) = \frac{e^{ah}-1}{ae^{ah}}e^{at}, \quad t \in T_h.$$

At small h, the functional form (6) of problem (4) is a "large" linear programming problem with a special matrix of basic constraints. It is well known that the solution of large special problems by conventional (general-purpose) methods is usually inefficient, because they do not take into account the special properties of such problems. Special implementations of general-purpose methods that take into account the specific properties of problems often make it possible to drastically enhance the efficiency of these methods. The method of potentials for solving transportation linear programming problems is a classical example [8]. In this paper, we present such a special method for solving problem (6).

In contrast to general linear programming problems, there is no need to store the problem parameters, in problem (6), since they can always be reconstructed from the parameters of the initial optimal control problem (4). In order to find $c(t)$ $(t \in T_h)$, it is sufficient to integrate the adjoint system

$$\dot{\psi}' = -\psi'A(t), \quad \dot{\psi}_{n+1} = a\psi_{n+1} - \psi'b(t) \tag{7}$$

with the initial conditions

$$\psi(t^*) = c, \quad \psi_{n+1}(t^*) = 0,$$

from left to right, and simultaneously calculate the integrals

$$c(t) = \int_t^{t+h} \psi_{n+1}(\vartheta)d\vartheta, \quad t \in T_h.$$

Similarly, in order to find the elements $d(t)$ $(t \in T_h)$, it is sufficient to integrate the homogeneous equation (2) from left to right, simultaneously calculating $\Phi(t)$, and multiply them on the right by the integral

$$\int_0^h \Phi^{-1}(\vartheta)d\vartheta$$

that is calculated only once.

In order to solve problem (6), we use the adaptive linear programming method described in [6], exploiting the "dynamic" nature of this problem. We will operate in terms of elements of the optimal control problem (4) or the adjoint system (7).

2.1. Support and Associated Elements

The basic instrument of the method described in [6] is the support. The pair $K_s = \{S_s, T_s\}$ consisting of the empty sets $S_s = \varnothing$ and $T_s = \varnothing$ is an (empty) support by definition. A nonepmty support $K_s = \{S_s, T_s\}$, $S_s \subset S_h$, $T_s \subset T_h$ has the following structure. The support set S_s consists of l^* nonoverlapping intervals $S_l = \{s_l, s_l + h, \ldots, s^l\}$. Each S_l is assigned an interval $T_l = \{t_l = s_l, t^l + h\ldots, t^l = s^l - h\}$ in T_s and an instance τ_l such that $s^{l-1} \le \tau_l \le s_l$. Thus,

$$S_s = \bigcup_{l=1}^{l^*} S_l, \quad T_s = \bigcup_{l=1}^{l^*} (T_l \cup \tau_l)$$

and every support of problem (4) has such a structure.

Since the intervals T_l are uniquely associated with S_l, below we interpret a support of problem (4) as a set of intervals S_l and times τ_l omitting T_l. For convenience, we assume that $s^0 = t_*$ and $s_{l^*+1} = t^*$.

The following components are associated with the support K_s:

(1) The function of potentials $\nu_h(s)$ $(s \in S_h)$:

$$\nu_h(s) = 0, \quad s \in S_{ns} = S_h \backslash S_s, \quad \nu_h(s) = \frac{c(s-h) - c(s)\Phi(h)}{d(h)}, \quad s \in S_l \backslash \{s_l, s^l\},$$

$$\nu_h(s_l) = \frac{c(\tau_l) - c(s_l)\Phi(s_l - \tau_l)}{d(s_l - \tau_l)}, \quad l = \overline{1, l^*},$$

$$\nu_h(s^l) = \frac{c(s^l - h) - c(\tau_{l+1})\Phi(\tau_{l+1} - s^l + h)}{d(h)}, \quad l = \overline{1, l^* - 1}, \quad \nu_h(s^{l^*}) = \frac{c(s^{l^*} - h)}{d(h)}.$$

Remark 3. If S_l contains a single point s_l $(l < l^*)$, the potential at this point is calculated by the formula

$$\nu_h(s_l) = \frac{c(\tau_l) - c(\tau_{l+1})\Phi(\tau_{l+1} - \tau_l)}{d(s_l - \tau_l)}.$$

(2) The cocontrol

$$\Delta_h(t) = c(t), \quad t \ge s^{l^*}, \quad \Delta_h(t) = c(t) - c(\tau_l)\Phi(\tau_l - t), \quad s^{l-1} \le t < s_l, \quad l = \overline{1, l^*}, \quad \Delta_h(t) = 0, \quad t \in T_s.$$

(3) The pseudosignal $\omega(t)$ $(t \in T)$ and the pseudocontrol $\zeta(s)$ $(s \in T)$. The nonsupport values of the pseudosignal and the support values of the pseudocontrol are

$$\omega(t) = M\,\mathrm{sgn}\,\Delta_h(t), \quad \text{if} \quad \Delta_h(t) \ne 0, \quad \omega(t) \in [-M, M], \quad \text{if} \quad \Delta_h(t) = 0, \quad t \in T_{ns} = T_h \backslash T_s,$$

$$\zeta(s) = L\,\mathrm{sgn}\,\nu_h(s), \quad \text{if} \quad \nu_h(s) \ne 0, \quad \zeta(s) \in [-L, L], \quad \text{if} \quad \nu_h(s) = 0, \quad s \in S_s.$$

The support values of the pseudosignal and pseudocontrol are calculated by the formulas

$$\omega(\tau_l) = \left[\zeta(s_l) - \Phi(s_l - s^{l-1})\zeta(s^{l-1}) - \sum_{s^{l-1} \le t < s_l,\, t \ne \tau_l} d(s_l - t)\omega(t)\right][d(s_l - \tau_l)]^{-1}, \quad \zeta(s^0) = u_0,$$

$$\omega(t) = \frac{\zeta(t+h) - \Phi(h)\zeta(t)}{d(h)}, \quad t \in T_l, \quad l = \overline{1, l^*},$$

$$\zeta(s) = \Phi(s - s^{l-1})\zeta(s^{l-1}) + \sum_{s^{l-1} \le t < s} d(s-t)\omega(t), \quad s^{l-1} < s < s_l, \quad l = \overline{1, l^* + 1}.$$

Let us describe a dynamic method for constructing the associated elements given the support K_s. Denote by $t_*(s) \in T_s (t_*(s) < s)$ the instance in T_s that is the nearest to s on the left, and by $t^*(s) \in T_s\,(t^*(s) \ge s)$ denote the instance that is the nearest to s on the right ($t^*(s) = s$ if $s \in T_s$). With regard for this notation and the equality $c(t) = \int_t^{t+h} \psi_{n+1}(\vartheta)d\vartheta$, we obtain the values of the potential function at the following support times:

$$\nu_h(s) = \left[\int_{t_*(s)}^{t_*(s)+h} \psi_{n+1}(\vartheta)d\vartheta - \int_{t^*(s)}^{t^*(s)+h} \psi_{n+1}(\vartheta)d\vartheta\Phi(t^*(s) - t_*(s))\right][d(s - t_*(s))]^{-1}, \quad s \in S_s,$$

and the values of the cocontrol at the nonsupport times

$$\Delta_h(t) = \int_t^{t+h} \psi_{n+1}(\vartheta)d\vartheta, \quad t \geq s^{l^*},$$

$$\Delta_h(t) = \int_t^{t+h} \psi_{n+1}(\vartheta)d\vartheta - \int_{\tau_l}^{\tau_l+h} \psi_{n+1}(\vartheta)d\vartheta\Phi(\tau_l - t), \quad s^{l-1} \leq t < s_l, \quad l = \overline{1, l^*}.$$

Hence, it is seen that in order to construct the potential and cocontrol functions given the support K_s, it is sufficient to integrate the adjoint system (7), Eq. (2) of the dynamic actuator, and store the values $\psi_{n+1}(\tau_l)$ and $\Phi(\tau_l - s_l)$ ($l = \overline{1, l^*}$). The second integration of system (7) and Eq. (2), and the stored quantities make it possible to calculate the functions $\nu(s)$ ($s \in S_h$) and $\Delta(t)$ $t \in T_h$.

2.2. The Maximum and ε-Maximum Principles

The optimality and suboptimality criteria for the feasible control signal and control, as well as the support optimality criterion, follow from the results obtained in [6].

Maximum principle. For the optimality of the feasible control signal $v(t)$ ($t \in T$) and control $u(t)$ ($t \in T$), it is necessary and sufficient that there exists a support K_s such that the following conditions are fulfilled for its associated elements:

(1) The maximum condition with respect to the control

$$\nu_h(s)u(s) = \max_{|u| \leq L} \nu_h(s)u, \quad s \in S_s;$$

(2) The maximum condition with respect to the control signal

$$\Delta_h(t)v(t) = \max_{|v| \leq M} \Delta_h(t)v, \quad t \in T_{ns}.$$

The support K_s for which the maximum principle holds is called the optimal support.

Support optimality criterion. For the optimality of the support K_s, it is necessary and sufficient that there exist associated elements $\zeta(s)$ ($s \in S$) and $\omega(t)$ ($t \in T$) such that

$$|\zeta(s)| \leq L, \quad s \in S_{ns}, \quad |\omega(t)| \leq M, \quad t \in T_s.$$

Then, $v^0(t) = \omega(t)$ ($t \in T_h$) and $u^0(s) = \zeta(s)$ ($s \in S$).

ε-Maximum principle. For any $\varepsilon \geq 0$, in order for the feasible control signal $v(t)$ ($t \in T$) and control $u(t)$ ($t \in T$) to be optimal, it is necessary and sufficient that there exists a support K_s such that the following relations are fulfilled for certain associated elements $\nu_h(s)$ ($s \in S_h$) and $\Delta(t)$ ($t \in T_h$):

(1) the ε-maximum condition with respect to the control:

$$\nu_h(s)u(s) = \max_{|u| \leq L} \nu_h(s)u - \varepsilon_u(s), \quad s \in S_s;$$

(2) The ε-maximum condition with respect to the control signal

$$\Delta_h(t)v(t) = \max_{|v| \leq M} \Delta_h(t)v - \varepsilon_v(t), \quad t \in T_{ns};$$

(3) The ε-accuracy condition

$$\sum_{s \in S_s} \varepsilon_u(s) + \sum_{t \in T_{ns}} \varepsilon_v(t) \leq \varepsilon.$$

2.3. The Dual Method for Solving the Maximum Excitation Problem

We construct the dual method in the form of an iterative process of sequential accumulation of optimal support intervals S_l ($l = \overline{1, l^*}$) based on the support replacement procedure involved in the adaptive method [6]. The method starts from the empty support and terminates when the support optimality criterion (see Subsection 2.2) is satisfied.

The algorithm has three phases. At the first phase, we find the support intervals S_l and the corresponding support times τ_l ($l = \overline{1, l^*}$). At the second phase, we refine the location of the endpoints of the intervals S_l; i.e., the location of the points s_l and s^l ($l = \overline{1, l^*}$). At the third phase, the times τ_l ($l = \overline{1, l^*}$) are refined.

In order to execute the iterations, it is required (as in [4]) to find nonsupport zeros of the cocontrol. The instance $t \in T_h \backslash t_*$ is called the nonsupport zero of the cocontrol if

$$\Delta_h(t-h)\Delta_h(t) < 0.$$

The set of all nonsupport zeros of the cocontrol located on the interval $[s^{l-1}, s_l]$ is denoted by $T^l_{\text{ns }0}$ ($l = \overline{1, l^* + 1}$). Given the instances $t \in T^l_0 = T^l_{\text{ns }0} \cup \{\tau_l, s^{l-1}, s_l\} = \{t^l_i, i = \overline{0, n^l + 1}\}$, which are the switching points of the control, and the numbers

$$\gamma^l = \operatorname{sgn}\Delta(s^{l-1}), \quad \text{if} \quad s^{l-1} \neq \tau_l, \quad \gamma^l = \operatorname{sgn}\Delta(s^{l-1} + h), \quad \text{if} \quad s^{l-1} = \tau_l, \quad l = \overline{1, l^* + 1},$$

one can reconstruct the values of the pseudocontrol at any time $t \in [s^{l-1}, s_l[$ by the rule $\omega(t) = (-1)^i\gamma^l M$, $t \in [t^l_{i-1}, t^l_i - h] \cap T_h$, $i = \overline{1, n^l + 1}$.

Assume that the following information is available from the computer memory at the beginning of any iteration step:

(a) the number of support intervals l^*;

(b) the endpoints s_l and s^l of the intervals S_l;

(c) the instances τ_l ($l = \overline{1, l^*}$);

(d) the potentials $\nu(s_l)$ and $\nu(s^l)$ ($l = \overline{1, l^*}$) at the endpoints of the support intervals;

(e) the pseudosignals $\omega(\tau_l)$ ($l = \overline{1, l^*}$);

(f) the pseudocontrols $\zeta_l = \zeta(s_l)$ on the support intervals S_l ($l = \overline{1, l^*}$) and the value $\zeta(t^*)$;

(g) the sets of nonsupport zeros of the cocontrol $T^l_{\text{ns }0}$ ($l = \overline{1, l^* + 1}$);

(h) the numbers γ^l ($l = \overline{1, l^* + 1}$);

(i) $\Phi(s_l - t)$, $t \in T^l_0 \backslash s_l$; $\psi_{n+1}(t)$, $t \in T^l_0$, $l = \overline{1, l^* + 1}$;

(j) the numbers

$$p_l = \sum_{s^{l-1} \le t < s_l,\, t \neq \tau_l} d(s_l - t)\omega(t), \quad l = \overline{1, l^*}.$$

The *first phase* starts from the empty support and consists in the sequential processing of each point $s_0 \in S_h$ beginning with the right endpoint ($s_0 = t^*$) of the interval S_h and ending with the left endpoint ($s_0 = t_* + h$) according to the following rules.

Stage 1. Calculate and store the pseudosignal $\zeta(t^*)$ (at the first iteration, $\zeta(s_0)$ is already available)

$$\zeta(s_0) = \Phi^{-1}(h)[\zeta(s_0 + h) - d(h)\omega(s_0)], \tag{8}$$

where $\omega(s_0) = (-1)^{i_0}\gamma^1 M$ for $s_0 \neq \tau_1$ and i_0 is such that $t^1_{i_0} \le s_0 < t^1_{i_0+1}$; for $s_0 = \tau_1$, the value $\omega(\tau_1)$ is taken from the computer memory.

Stage 2. If $|\zeta(s_0)| \le L$, then go to Stage 5.

Stage 3. If $|\zeta(s_0)| > L$ and $s_0 < s_1 - h$ (this includes the first iteration at $s_0 = t^*$), then initiate the generation of a new interval.

Stage 4. If $|\zeta(s_0)| > L$ and $s_0 = s_1 - h$ (this cannot occur at the first iteration), then extend or move the first support interval to the left.

Stage 5. Set $s_0 := s_0 - h$, and repeat Stages 1–5 until all the points of the interval S_h are processed.

To execute Stages 3 and 4, the long dual step σ^* (see [6]) must be calculated; this step is found as a minimizer of the objective function $\phi(\sigma)$ ($\sigma \ge 0$) of the problem that is dual to (4) along the varied potential function and cocontrol

$$\nu(s;\sigma) = \nu_h(s) + \sigma\Delta\nu(s), \quad s \in T_h, \quad \delta(t;\sigma) = \Delta_h(t) + \sigma\Delta\delta(t), \quad t \in T_h.$$

Here, the variation $\Delta\nu(s)$ ($s \in S_h$) of the potential function and the variation $\Delta\delta(t)$ ($t \in T_h$) of the cocontrol are constructed depending on two subcases:

$$\text{(I) } \tau_1 < s_0 < s_1, \quad \text{(II) } s_0 \le \tau_1,$$

by the following rules:

$$\text{(I) } \Delta\nu(s_0) = \operatorname{sgn}\zeta(s_0), \quad \Delta\nu(s_1) = -\Delta\nu(s_0)\Phi(s_0 - s_1), \quad \Delta\nu(s) = 0, \quad s \in S_h \backslash \{s_0, s_1\},$$
$$\Delta\delta(t) = \Delta\nu(s_0)d(s_0 - t), \quad s_0 \le t < s_1, \quad \Delta\delta(t) = 0, \quad t < s_0, \quad t \ge s_1;$$

$$\text{(II) } \Delta\nu(s_0) = \operatorname{sgn}\zeta(s_0), \quad \Delta\nu(s) = 0, \quad s \in S_h \backslash s_0, \quad \Delta\delta(t) = \Delta\nu(s_0)d(s_0 - t), \quad t < s_0,$$
$$\Delta\delta(t) = 0, \quad t \ge s_0.$$

It was shown in [6] that the function $\phi(\sigma)$ ($\sigma \ge 0$) decreases with the rate

$$\alpha^1 = -\rho(\zeta(s_0), [-L, L]),$$

in a small right-hand neighborhood of the point $\sigma = 0$; here, $\rho(c, [a, b])$ is the distance from the point c to the interval $[a, b]$. In addition, $\phi(\sigma)$ ($\sigma \ge 0$) is piecewise linear with the nodes at the values of σ at which the function $\nu(s; \sigma)$ has a new zero in s or the function $\delta(t; \sigma)$ has the new zero in t. These values are calculated by the formulas

$$\sigma_\nu(s) = \begin{cases} -\nu_h(s)/\Delta\nu(s), & \nu_h(s)\Delta\nu(s) < 0, \\ \infty, \quad \nu_h(s)\Delta\nu(s) \ge 0, & s \in S_s, \end{cases}$$
$$\sigma_\delta(t) = \begin{cases} -\Delta_h(t)/\Delta\delta(t), & \Delta_h(t)\Delta\delta(t) < 0, \\ \infty, \quad \Delta_h(t)\Delta\delta(t) \ge 0, & t \in T_{\text{ns}}. \end{cases} \tag{9}$$

Assume that all nonzero steps $\sigma_\nu(s)$ and $s \in S_s$, $\sigma_\delta(t)$, $t \in T_{\text{ns}}$ are different and arranged in ascending order $0 < \sigma^1 < \ldots < \sigma^{k^0}$, α^k is the rate of variation of the dual objective function on the interval $[\sigma^{k-1}, \sigma^k[$ ($\sigma^0 = 0$). For the rate α^{k+1} on the interval $[\sigma^k, \sigma^{k+1}[$, the following formula holds:

$$\alpha^{k+1} = \alpha^k + \Delta\alpha^k, \quad \Delta\alpha^k = \begin{cases} 2M|\Delta\delta(t_k)|, & \sigma^k = \sigma_\delta(t_k), \\ 2L|\Delta\nu(s_k)|, & \sigma^k = \sigma_\nu(s_k). \end{cases}$$

The number $\sigma^* = \sigma^{k^*}$ such that $\alpha^{k^*} < 0$ and $\alpha^{k^*+1} \ge 0$ is called the long dual step (see [6]).

In order to construct a fast and efficient algorithm for solving problem (4), it is first required to calculate the long dual step σ^* quickly and efficiently. In problem (4), this is possible due to two reasons.

First, it is clear that, as $\sigma \ge 0$ increases, new zeros of the function $\delta(t; \sigma)$ ($t \in T_h$) appear in the neighborhood of nonsupport zeros of the cocontrol $\Delta_h(t)$ ($t \in T_h$) thus representing the motion of these zeros along the time axis. This fact and the data (i) stored in computer memory, as well as the value $\Phi(s_0 - s_1)$, which can be easily calculated at Stage 1, make it possible to rapidly calculate the steps $\sigma_\delta(t)$. Such a procedure was described in detail in [4] for the terminal direct optimal control problem in the class of discrete controls. Since the variation of the cocontrol $\Delta\delta(t)$ is distinct from zero only on the interval T_1, the application of the operations described in [4] to the nonsupport zeros in $T^l_{\text{ns}\,0}$ allows one to quickly calculate the long dual step σ^*.

Second, it follows from (9) that among the steps $\sigma^\nu(s)$ $(s \in S_s)$, only in case (I), under the condition $\Delta\nu(s_0) = \operatorname{sgn}\nu_h(s_1)$, just one finite step can be realized (all other steps are infinite). In all other cases, we have the identity $\Delta\nu(s) = 0$ $(s \in S_s)$, which implies that $\sigma_\nu(s) = \infty$ for $s \in S_s$. This means that most of the steps need not be calculated. Therefore, the excitation of the potential function at the point s_0 does not propagate to the support time instances that lie to the right of s_1 for case (I) and to the right of τ_1 for case (II). This fundamental property of the maximum excitation problem saves us the computationally costly (for small sampling periods h) processing of phase constraints and makes the cost of the method of solving problem (4) equivalent to that of the method [4] designed for the terminal problem without phase constraints.

Assume that the long dual step σ^* is already calculated. Two situations are possible: (a) $\sigma^* = \sigma_\delta(\tau_*)$ and (b) $\sigma^* = \sigma_\nu(s_*)$. Let us describe the rules of constructing the new support $\bar{K}_s$for Stages 3 and 4.

For Stage 3, in situation (a) we rearrange the intervals of the old support $\bar{S}_{l+1} = S_l$ $(l = \overline{1, l^*})$, set $\bar{l}^* = l^* + 1$, and introduce the new interval $\bar{S}_1 = \{s_0\}$ into the support. The rules for constructing support instances are different for cases (I) and (II):

$$\text{(I)}\ \bar{\tau}_1 = \tau_1, \quad \bar{\tau}_2 = \tau_*, \quad \bar{\tau}_{l+1} = \tau_l, \quad l = \overline{2, l^*};$$

$$\text{(II)}\ \bar{\tau}_1 = \tau_*, \quad \bar{\tau}_{l+1} = \tau_l, \quad l = \overline{1, l^*}.$$

The situation (b) can be realized only in case (I). The new support is

$$\bar{l}^* = l^* + 1, \quad \bar{S}_1 = \{s_0\}, \quad \bar{S}_2 = S_1 \backslash s_1, \quad \bar{S}_{l+1} = S_l, \quad l = \overline{2, l^*},$$

$$\bar{\tau}_1 = \tau_1, \quad \bar{\tau}_2 = s_1, \quad \bar{\tau}_l = \tau_l, \quad l = \overline{2, l^*}.$$

For Stage 4, in situation (a) the time s_0 is included in the support; more precisely, it is added to the support interval on the left: $\bar{S}_1 = S_1 \cup s_0$. In case (I) (obviously, $\tau_* = s_0$ in this case), the support time corresponding to this interval does not change: $\bar{\tau}_1 = \tau_1$. The time τ_* at which the long dual step is realized becomes a support time in case (II): $\tau_1 = \tau_*$.

The situation (b) at Stage 4 is possible only in case (I) under the condition that the first interval of the old support consists of a single point: $S_1 = \{s_1\}$. Then, we move the first interval to the left and replace the time s_1 with s_0: $\bar{S}_1 = s_0$.

In addition to changing the support, one must modify the data (d)–(j) so that they can be used at the next iteration step. The sets $\bar{T}^l_{\text{ns}\,0}$, the numbers $\bar{\gamma}^l$, $\bar{p}_l$, and $\psi(t)$ $(t \in \bar{T}^l_0, l = \overline{1, l^* + 1})$ are calculated in the course of the calculation of the long dual step using the rules presented in [4]. The values $\Phi(\bar{s}_1 - t)$ $(t \in \bar{T}^l_0)$ are obtained by multiplying the old values $\Phi(s_1 - t)$ $(t \in T^1_0)$ by $\Phi(s_0 - s_1)$. The potential function at the point $\bar{s}_1$ is found by the rule $\nu(\bar{s}_1) = \sigma^*\Delta\nu(s_0)$. The pseudocontrols on the interval $\bar{S}_1$ are set to $\bar{\zeta}_1 = \Delta\nu(s_0)L$. The pseudosignal at the time $\bar{\tau}_1$, and for $\bar{\tau}_2 \neq \tau_2$ also at the time τ_2, is calculated by the formulas

$$\omega(\bar{\tau}_1) = \frac{\zeta(\bar{s}_1) - \Phi(\bar{s}_1 - t_*)u_0 - \bar{p}_1}{d(\bar{s}_1 - \bar{\tau}_1)}, \quad \omega(\bar{\tau}_2) = \frac{\zeta(\bar{s}_2) - \Phi(\bar{s}_2 - \bar{s}^1)\bar{\zeta}_1 - p_2}{d(\bar{s}_2 - \bar{\tau}_2)}.$$

The first phase is completed by processing the leftmost time $t_* + h$. The first phase yields a set of support intervals S_l and the corresponding times τ_l $(l = \overline{1, l^*})$.

The *second phase* consists of processing the endpoints of the support intervals S_l $(l = \overline{1, l^*})$ produced at the first phase. These are the points $s_1 - h, s^1 + h, \ldots, s_{l^*} - h, s^{l^*} + h, t^*$ at which the constraints can be violated. The elimination of such overshoots is done according to the following rules:

Stage 1. Set $l_0 = l^*$, and $s_0 = s_{l_0} - h$.

Stage 2. Calculate $\zeta(s_0)$ by formula (8).

Stage 3. If $|\zeta(s_0)| > L$, then extend the support interval S_{l_0} to the left (as at Stage 4 of the first phase), set $s_0 := s_0 - h$, and return to Stage 2.

Stage 4. If $|\zeta(s_0)| \le L$, then set $s_0 = s^{l_0 - 1} + h$.

Stage 5. Calculate $\zeta(s_0)$ by the formula

$$\zeta(s_0) = \Phi(h)\zeta_{l_0 - 1} + d(h)\omega(s^{l_0 - 1}),$$

where $\omega(s^{l_0 - 1}) = \gamma_{l_0} M$ if $s^{l_0 - 1} \neq \tau_{l_0}$ and $\omega(\tau_{l_0})$ does not change if $s^{l_0 - 1} = \tau_{l_0}$.

Stage 6. If $|\zeta(s_0)| > L$ then extend the support interval S_{l-1} to the right, set $s_0 := s_0 + h$, and return to Stage 5.

Stage 7. Repeat Stages 2–6 until the condition $|\zeta(s)| \le L$, $(s \in [s^{l_0 - 1}, s_{l_0}]$ is satisfied.

Stage 8. Set $l_0 := l_0 - 1$, $s_0 := s_{l_0} - h$, repeat Stages 2–7 and, if necessary, Stages 1–7 until all overshoots are eliminated.

The execution of Stages 3–6 begins with the construction of the variations $\Delta\nu(s)$ $(s \in S_h)$ and $\Delta\delta(t)$ $(t \in T_h)$ for cases (I) for $\tau_{l_0} < s_0 < s_{l_0}$ and (II) for $s^{l_0 - 1} < s_0 \le \tau_{l_0}$:

$$\text{(I) } \Delta\nu(s_0) = \operatorname{sgn}\zeta(s_0), \quad \Delta\nu(s_{l_0}) = -\Delta\nu(s_0)\Phi(s_0 - s_{l_0}), \quad \Delta\nu(s) = 0, \quad s \in S_h \backslash \{s_0, s_{l_0}\},$$
$$\Delta\delta(t) = \Delta\nu(s_0)d(s_0 - t), \quad s_0 \le t < s_{l_0}, \quad \Delta\delta(t) = 0, \quad t < s_0, \quad t \ge s_{l_0};$$

$$\text{(II) } \Delta\nu(s_0) = \operatorname{sgn}\zeta(s_0), \quad \Delta\nu(s^{l_0 - 1}) = -\Delta\nu(s_0)\Phi(s_0 - s^{l_0 - 1}), \quad \Delta\nu(s) = 0, \quad s \in S_h \backslash \{s_0, s^{l_0 - 1}\},$$
$$\Delta\delta(t) = \Delta\nu(s_0)d(s_0 - t), \quad s^{l_0 - 1} \le t < s_0, \quad \Delta\delta(t) = 0, \quad t < s^{l_0 - 1}, \quad t \ge s_0.$$

Then, as at the first phase, the long dual step σ^* is calculated, and the support is modified according to the following rules.

At Stage 3, in case (I), we set $\bar{S}_l = S_l$ $(l = \overline{1, l^*}, l \neq l_0)$ and $\bar{\tau}_l = \tau_l$ $(l = \overline{1, l^*})$. The support interval $\bar{S}_{l_0}$ is constructed depending on which situation (a) or (b) was realized in the course of calculation of the long dual step: (a) $\bar{S}_{l_0} = S_{l_0} \cup s_0$; (b) (can be realized only if $S_{l_0} = \{s_{l_0}\}$) $\bar{S}_{l_0} = \{s_0\}$.

In case (II) and situation (a), the new support is constructed by the rule $\bar{S}_l = S_l$, $\bar{\tau}_l = \tau_l$ $(l = \overline{1, l^*}, l \neq l_0)$ and $\bar{S}_{l_0} = S_{l_0} \cup s_0$, $\bar{\tau}_{l_0} = \tau_*$. The situation (b) is possible only if $\operatorname{sgn}\nu(s^{l_0 - 1}) = \Delta\nu(s_0)$. Then, the new support $\bar{K}_s$ has the form $\bar{S}_l = S_l$, $\bar{\tau}_l = \tau_l$ $(l = \overline{1, l_0 - 2}, l = \overline{l_0 + 1, l^*})$ and $\bar{S}_{l_0 - 1} = S_{l_0 - 1} \backslash s^{l_0 - 1}$, $\bar{S}_{l_0} = S_{l_0} \cup s_0$, $\bar{\tau}_{l_0} = s^{l_0 - 1}$.

At Stage 6, in case (Ia), we set $\bar{S}_{l_0 - 1} = S_{l_0 - 1} \cup s_0$, $\tau_{l_0} = \tau_*$, and leave the other support intervals and times intact. If $\operatorname{sgn}\nu(s_{l_0}) = \Delta\nu(s_0)$, then the situation (b) can be realized in which two intervals are changed: $\bar{S}_{l_0 - 1} = S_{l_0 - 1} \cup s_0$, $\bar{S}_{l_0} = S_{l_0} \backslash s_{l_0}$, and the working support time $\bar{\tau}_{l_0} = s_{l_0}$.

In situation (IIa), we extend the interval $S_{l_0 - 1}$: $\bar{S}_{l_0 - 1} = S_{l_0 - 1} \cup s_0$, and in situation (IIb), we move the interval consisting of the single point $s_{l_0 - 1}$ to the point s_0: $\bar{S}_{l_0 - 1} = \{s_0\}$.

The second phase completes when the values of the pseudosignal are within the interval $[-L, L]$ for all points $s_1 - h, s^1 + h, \ldots, s_{l^*} - h, s^{l^*} + h, t^*$.

At the *third phase*, we refine the location of the support times τ_l ($l = \overline{1, l^*}$) by executing iterations for every τ_l such that $|\omega(\tau_l)| > M$. Variations of the functions $\nu(s)$ ($s \in S_h$) and $\Delta(t)$ ($t \in T_h$) are constructed by the rules

$$\Delta\delta(\tau_l) = \operatorname{sgn}\omega(t_l), \quad \Delta\delta(t) = \Delta\delta(\tau_l)\Phi(\tau_l - t), \quad s^{l-1} - h \le t < s_l,$$

$$\Delta\delta(t) = 0, \quad t < s^{l-1} - h, \quad t \ge s_l, \quad \Delta\nu(s) = 0, \quad s \in S_h \backslash \{s_l, s^{l-1}\};$$

$$\Delta\nu(s_l) = -\Delta\delta(\tau_l)/d(s_l - \tau_l), \quad \Delta\nu(s^{l-1}) = \Delta\delta(\tau_l)/d(s^{l-1} - \tau_l).$$

The simplest case of the support change occurs when situation (a) ($\sigma^* = \sigma_\delta(\tau^*)$) is realized in the course of the long dual step calculation. Then, only the current working support time is changed: $\bar{\tau}_l = \tau_*$. In situation (b), under the condition $s_* = s_l$, we set $\bar{\tau}_l = s_l$ and $\bar{S}_l = \bar{S}_l \backslash s_l$. If $s_* = s^{l-1}$, then $\bar{\tau}_l = s_l$ and $\bar{S}_l = S_l \backslash s_l$. If $s_* = s^{l-1}$, then $\bar{\tau}_l = s_l$ and $\bar{S}_l = S_l \backslash s_l$.

The third phase completes when the condition $|\omega(\tau_l)| \le M$ ($l = \overline{1, l^*}$) is fulfilled.

Remark 4. Upon completing all three phases, the support optimality criterion can be still violated at a small number of points. Then, Phases 2 and 3 must be repeated.

2.4. Example

To illustrate the algorithm described in Subsection 2.3, we construct the optimal control based on a dynamic actuator in the maximum acceleration problem for an oscillator that is in equilibrium at the initial time $t_* = 0$:

$$\begin{gathered} x_2(12) \longrightarrow \max, \\ \dot{x}_1 - x_2, \quad \dot{x}_2 = -x_1 + u, \quad x_1(0) = x_2(0) = 0, \\ \dot{u} = -u + v, \quad u(0) = 0, \\ |u(t)| \le 0.5, \quad |v(t)| \le 1, \quad t \in T = [0, 12]. \end{gathered} \tag{10}$$

Problem (10) was solved in the class of discrete control signals $v(t)$ ($t \in T$) with the sampling period $h = 0.1$. Figure 1a shows the pseudocontrol $\zeta(s)$ ($s \in S_h$) associated with the initial empty support. The first phase gave four support intervals

$$S_1 = [1.7, 3.6], \quad S_2 = [5.0, 6.6], \quad S_3 = [9.0, 9.9], \quad S_4 = [11.3, 12.0],$$

with the corresponding times

$$\tau_1 = 1.6, \quad \tau_2 = 4.9, \quad \tau_3 = 8.9, \quad \tau_4 = 11.2.$$

The pseudocontrol $\zeta(s)$ ($s \in S$) is shown in Fig. 1b. It is seen that there are overshoots of the pseudocontrol beyond the limits of the interval [–0.5, 0.5] to the left of the intervals S_2, S_3, and S_4. These overshoots are eliminated at the second phase (Fig. 1c). At this phase, the following values of the pseudosignals $\omega(t)$ were obtained at the support times τ_l ($l = \overline{1, 4}$):

$$\omega(\tau_1) = -0.821974599, \quad \omega(\tau_2) = 0.992713787, \quad \omega(\tau_3) = -0.98019402, \quad \omega(\tau_4) = 0.992713787.$$

These values are within the interval [–1, 1]; therefore, the third phase was not needed. Figure 1c shows the optimal open-loop control in problem (10).

The optimal support K_s^0 in problem (10) is

$$S_1^0 = [1.7, 3.7], \quad S_2^0 = [4.8, 6.8], \quad S_3^0 = [7.9, 10.0], \quad S_4^0 = [11.1, 12.0],$$

$$\tau_1^0 = 1.6, \quad \tau_2^0 = 4.7, \quad \tau_3^0 = 6.8, \quad \tau_4^0 = 11.0.$$

The cocontrol $\Delta_h^0(t)$ ($t \in T_h$) associated with the optimal support K_s^0 is shown in Fig. 2. It has a single nonsupport zero at $t = 0.6$. The phase trajectory on the plane x_1x_2 is shown in Fig. 3.

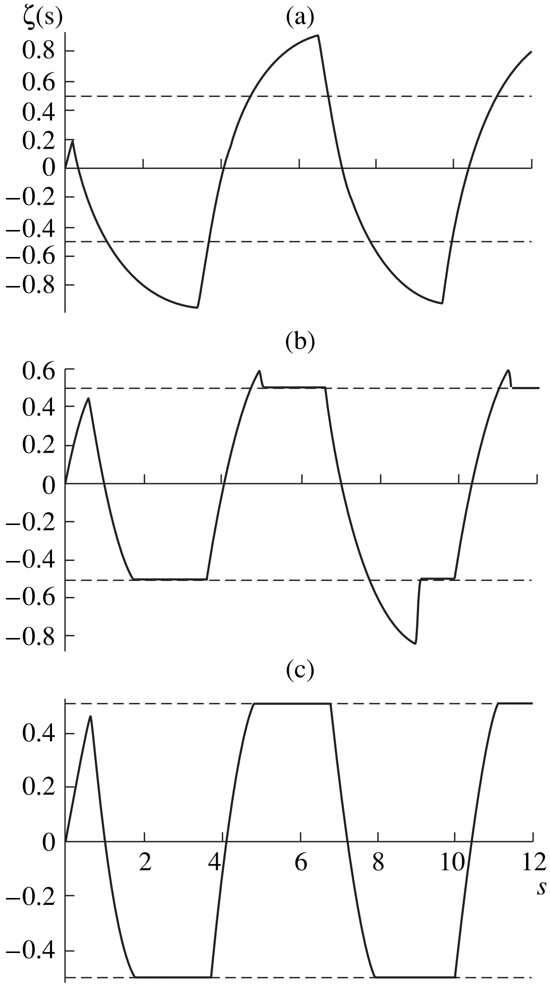

Fig. 1.

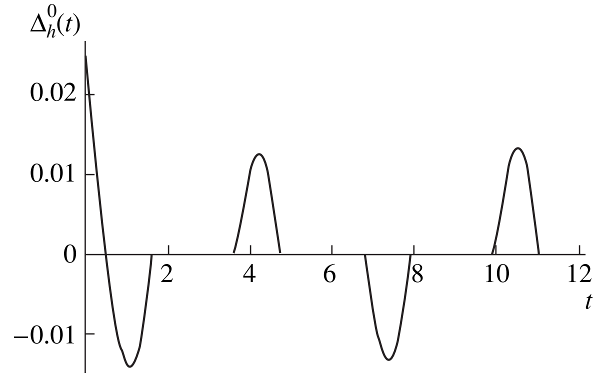

Fig. 2.

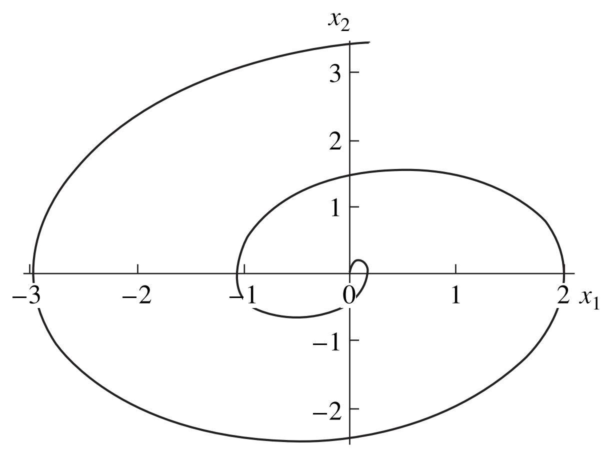

Fig. 3.

At the terminal time $t^* = 12$, system (10) is at the state $x_1^0(12) = = 0.1852543971$, $x_2^0(12) = 3.437540698$; i.e., the objective function is 3.437540698.

2.5. Refinement of the Discrete Control to Make Them Piecewise Constant

A specific feature of the method described in Subsection 2.3 is that it produces the control that changes its values (switches) only at discrete times $t \in T_h$.

We describe a refinement procedure that constructs an open-loop solution to problem (4) in the class of piecewise constant controls with arbitrary switching points on the interval T.

Assume that, for a small sampling period h, the optimal support $K_s^0(h)$ with the support intervals $S_l^0(h) = [s_l^0(h), s^{0l}(h)]$ and the working support times $\tau_l^0(h)$ $(l = \overline{1, l^*})$ is already constructed. Define

$$K_s^0 = \lim_{h \to 0} K_s^0(h): s_l^0 = \lim_{h \to 0} s_l^0(h), \quad s^{0l} = \lim_{h \to 0} s^{0l}(h), \quad S_l^0 = [s_l^0, s^{0l}].$$

The refinement procedure consists of solving a special system of refinement equations by the Newton method. To set up the refinement equations, we use continuous analogs of the potential function $\nu_h(s)$ $(s \in S_h)$ and cocontrol $\Delta_h(t)$ $(t \in T_h)$.

Define the piecewise continuous function $\lambda(t)$ ($t \in T$) with discontinuities at the points s_l and s^l ($l = \overline{1, l^*}$):

$$\lambda(t) = 0, \quad t \in [s_l, s^l], \quad l = \overline{1, l^*}; \tag{11}$$

$\lambda(t)$ for $t \in\ ]s^{l-1}, s_l[$, $l = \overline{1, l^* + 1}$ is the solution to the adjoint equation

$$\dot{\lambda} = -a\lambda - \psi_c' b(t) \tag{12}$$

with the initial conditions

$$\lambda(\tau_l) = 0, \quad l = \overline{1, l^*}, \quad \lambda(t^*) = 0. \tag{13}$$

Here, $\psi_c(t)$ ($t \in T$) is the solution to the adjoint equation $\dot{\psi}' = -\psi' A(t)$, $\psi(t^*) = c$.

It is easy to show that the functions $\nu_h(s)$ ($s \in S_h$) and $\Delta_h(t)$ ($t \in T_h$) have the following properties:

$$\nu_h(s)/h \longrightarrow \psi_c'(s)b(s), \quad s \in S_l \backslash \{s_l, s^l\}, \quad \nu_h(s_l) \longrightarrow -\lambda(s_l - 0), \quad \nu_h(s^l) \longrightarrow \lambda(s^l + 0), \quad h \longrightarrow 0,$$

$$\Delta_h(t)/h \longrightarrow \lambda(t), \quad h \longrightarrow 0.$$

We show that the function $\lambda(t)$ ($t \in T$) constructed by rules (11)–(13) is continuous on the support K_s^0. To prove this fact, we use the optimality conditions, which imply that if the optimal control takes the value at the upper bound ($u(s_l^0(h)) = L$) then this bound is reached when the control signal is positive $v^0(s_l^0(h) - 0) = M$ or $v^0(s_l^0(h) - 0) = \omega(s_l^0(h) - h)$ if $\tau_l^0 = s_l - h$. Then, the inequalities $\nu_h^0(s_l^0(h)) \geq 0$ and $\Delta_h^0(s_l^0(h) - h) \geq 0$ hold. Letting h tend to zero, we obtain

$$\nu_h^0(s_l^0(h)) \longrightarrow -\lambda(s_l^0 - 0) \geq 0, \quad \Delta_h^0(s_l^0(h) - h)/h \longrightarrow \lambda(s_l^0 - 0) \geq 0, \quad h \longrightarrow 0,$$

which is possible only when $\lambda(s_l^0 - 0) = 0$. A similar conclusion is obtained for $u(s_l^0(h)) = -L$ and $v(s_l^0(h) - 0) \leq 0$.

A similar reasoning applied to the point $s^{0l-1}(h)$ yields $\lambda(s^{0l-1} + 0) = 0$.

Therefore, the function $\lambda(t)$ ($t \in T$) has the property

$$\lambda(s_l^0) = \lambda(s^{0l}) = 0, \quad l = \overline{1, l^*}, \tag{14}$$

at the endpoints of the support intervals S_l^0 and is continuous at these points. This property explains the fact that the working support times $\tau_l^0(h)$ in numerical examples, including the example considered in Subsection 2.4 (see Fig. 1b), are located near the endpoints of the support intervals: $\tau_l^0(h) = s_l^0(h) - h$ or $\tau_l^0(h) = s^{0l-1}(h)$. As $h \longrightarrow 0$, they stick either to the left endpoint of the interval S_l^0 or to the right endpoint of S_{l-1}^0; therefore, they actually do not take part in switching the control.

The structure of a piecewise constant control signal is defined as the set of its switching points

$$t_1^1, \ldots, t_{n^1}^1, s_1, s^1, t_1^2, \ldots, t_{n^2}^2, s_2, \ldots, s^{l-1}, t_1^l, \ldots, t_{n^l}^l, s_l, s^l, \ldots, s^{l^*}, t_1^{l^*+1}, \ldots, t_{n^{l^*+1}}^{l^*+1}, \tag{15}$$

the numbers γ^l, which determine the signs of the control signal on the intervals $[s^{l-1}, t_1^l[$ ($l = \overline{1, l^* + 1}$), and the numbers $u_l = (-1)^{n^l}\gamma^l L$, which determine the values of the controls on the intervals $[s_l, s^l]$. The function $v(t)$ ($t \in T$) can be uniquely reconstructed from these data. For convenience, we set $t_0^l = s^{l-1}$ and $t_{n^l+1}^l =$

s_l ($l = \overline{1, l^* + 1}$). Then, the control signal can be written in the form

$$v(t) = \begin{cases} (-1)^i \gamma^l M, & t \in [t_i^l, t_{i+1}^l[, \quad i = \overline{0, n^l}, \quad l = \overline{1, l^* + 1}, \\ -au_l, & t \in [s_l, s^l[, \quad l = \overline{1, l^*}. \end{cases} \tag{16}$$

At t_i^l ($i = \overline{1, n^l}$, $l = \overline{1, l^* + 1}$), the control signal switches from $v(t_i^l - 0) = M$ to $(t_i^l + 0) = -M$ or, conversely, from $v(t_i^l - 0) = -M$ to $v(t_i^l + 0) = M$. At the instances s_l, the control signal switches from $v(s_l - 0) = M$ to $v(s_l) = -aL$ or from $v(s_l - 0) = -M$ to $v(s_l) = aL$; at the instances s^l from $v(s^l) = aL$ to $v(s^l + 0) = -M$ or from $v(s^l) = aL$ to $v(s^l + 0) = M$

Let us set up equations relating variables (15). The first group of equations is obtained by representing $u(s_l) = u_l$ in terms of $u(s^{l-1}) = u_{l-1}$ on the basis of the Cauchy formula:

$$u_l = \Phi(s_l - s^{l-1})u_{l-1} + \sum_{i=0}^{n^l} (-1)^i \gamma^l M \int_{t_i^l}^{t_{i+1}^l} \Phi(s_l - \vartheta) d\vartheta, \quad l = \overline{1, l^*}. \tag{17}$$

The second group of equations follows from the maximum principle, which claims that the switching points t_i^l ($i = \overline{1, n^l}$, $l = \overline{1, l^* + 1}$) are zeros of the function $\lambda(t)$ ($t \in T$):

$$\lambda(t_i^l) = \int_{t_i^l}^{s_l} \psi_c'(\vartheta) b(\vartheta) \Phi(\vartheta - t_i^l) d\vartheta = 0, \quad i = \overline{1, n^l}, \quad l = \overline{1, l^* + 1}. \tag{18}$$

The last group of equations connects the points s_l and s^{l-1} by property (14):

$$\lambda(s^{l-1}) = \int_{s^{l-1}}^{s_l} \psi_c'(\vartheta) b(\vartheta) \Phi(\vartheta - s^{l-1}) d\vartheta = 0, \quad l = \overline{2, l^* + 1}. \tag{19}$$

The system of equations (17)–(19) decomposes with respect to the variables

$$y^1 = (t_1^1, \ldots, t_{n^1}^1, s_1), \quad y^l = (s^{l-1}, t_1^l, \ldots, t_{n^l}^l, s_l), \quad l = \overline{2, l^*}, \quad y^{l^*+1} = (s^{l^*}, t_1^{l^*+1}, \ldots, t_{n^{l^*+1}}^{l^*+1}).$$

The unknowns y^l ($l = \overline{1, l^* + 1}$) can be found from (17)–(19) for every l.

System (17)–(19) can be written in the form

$$R_l(y^l) = \begin{pmatrix} \displaystyle\int_{s^{l-1}}^{s_l} \psi_c'(\vartheta) b(\vartheta) \Phi(\vartheta - s^{l-1}) d\vartheta \\ \displaystyle\int_{t_i^l}^{s_l} \psi_c'(\vartheta) b(\vartheta) \Phi(\vartheta - t_i^l) d\vartheta, \quad i = \overline{1, n^l} \\ \displaystyle\Phi(s_l - s^{l-1})u_{l-1} + \sum_{i=0}^{n^l} (-1)^i \gamma^l M \int_{t_i^l}^{t_{i+1}^l} \Phi(s_l - \vartheta) d\vartheta - u_l \end{pmatrix} = 0, \quad l = \overline{1, l^* + 1}. \tag{20}$$

The equation $R_1(y^1) = 0$ does not contain the equation from group (19), and the equation $R_{l^*+1}(y^{l^*+1}) = 0$ does not contain the equation from group (17).

The Jacobians $I_l(y^l)$ of the system $R_l(y^l) = 0$ have the form

$$I_1(y^1) = \begin{pmatrix} A_{22}^1 & A_{23}^1 \\ A_{32}^1 & A_{33}^1 \end{pmatrix}, \quad I_l(y^l) = \begin{pmatrix} A_{11}^l & 0_{1\times n^l} & A_{13}^l \\ 0_{n^l\times 1} & A_{22}^l & A_{23}^l \\ A_{31}^l & A_{32}^l & A_{33}^l \end{pmatrix}, \quad I_{l^*+1}(y^{l^*+1}) = \begin{pmatrix} A_{11}^{l^*+1} & 0_{1\times n^{l^*+1}} \\ 0_{n^{l^*+1}\times 1} & A_{22}^{l^*+1} \end{pmatrix}, \tag{21}$$

where $A_{11}^l = [\dot\lambda(s^{l-1})]$, $A_{13}^l = [\psi'_c(s_l)b(s_l)\Phi(s_l - s^{l-1})]$, $A_{22}^l = \mathrm{diag}(\dot\lambda(t_i^l), i = \overline{1, n^l})$, $A_{23}^l = [\psi'_c(s_l)b(s_l)\Phi(s_l - t_i^l), i = \overline{1, n^l}]'$, $A_{31}^l = [-\Phi(s_l - s^{l-1})(au_{l-1} + \gamma^l M)]$, $A_{32}^l = [-2M\gamma^l(-1)^i\Phi(s_l - t_i^1), i = \overline{1, n^l}]$, and $A_{33}^l = [au_l + (-1)^{n^l}M\gamma^l]$.

It is easy to verify that, if the condition $\dot\lambda(t_i^l) \neq 0$ $(i = \overline{0, n^l})$ is satisfied for all $l = \overline{1, l^* + 1}$, the matrices $I_l(y^l)$ are nonsingular. Therefore, system (20) can be solved by the Newton method using the values $s^{0l-1}(h)$, $t_i^{0l}(h)$ $(i = \overline{1, n^l})$, and $s_l^0(h)$, which are found by the method described in Subsection 2.3, as the initial approximations for small h.

At Newton iterations, it is required to quickly calculate $R_l(y^l)$ and $I_l(y^l)$. When solving problem (4) in the class of discrete control signals, we calculate and store the integrals

$$\int_{t_i^{0l}(h)}^{s_l^0(h)} \psi'_c(\vartheta)b(\vartheta)\Phi(\vartheta - s^{0l-1}(h))d\vartheta, \quad i = \overline{0, n^{0l}}, \quad l = \overline{1, l^* + 1}. \tag{22}$$

Then, the first $n^l + 1$ components of $R_l(y^l)$ can be calculated at the first iteration of the Newton method. The last component is obtained from the known number p_l by the formula

$$\sum_{i=0}^{n^l} (-1)^i\gamma^l M \int_{t_i^{0l}(h)}^{t_{i+1}^{0l}(h)} \Phi(s_l - \vartheta)d\vartheta = p_l + (-1)^{i_\tau}\gamma^l M \int_{\tau_l^0(h)}^{\tau_l^0(h) + h} \Phi(s_l - \vartheta)d\vartheta, \tag{23}$$

where i_τ is the index of the time $\tau_l^0(h)$ in the set $T_0^{0l}(h)$.

To quickly find the matrices $I_l(y^l)$, in addition to quantities (i) stored in the computer memory, we also store $\psi_c(t)$ $(t \in T_0^{0l}(h), l = \overline{1, l^* + 1})$. This enables us to find $\dot\lambda(t_i^{0l}(h)) = -a\lambda(t_i^{0l}(h)) - \psi'_c(t_i^{0l}(h))b(t_i^{0l}(h))$ $(i = \overline{0, n^l}, l = \overline{1, l^* + 1})$, which are involved in A_{11} and A_{12}, and elements of the matrices A_{13}^l and A_{23}^l. To recalculate integrals (22), (23) and the data (i) stored in the memory at Newton iterations, the integration of the corresponding equations is continued from the old to the new values of the unknowns.

Example 1. The refinement procedure described above was used to solve problem (10) in the class of piecewise constant controls. As a result, the following refined endpoints of the intervals S_l^0 were obtained:

$$S_1^0 = [1.623895502, 3.697213951], \quad S_2^0 = [4.795826239, 6.838806604],$$

$$S_3^0 = [7.937418893, 9.980399258], \quad S_4^0 = [11.07901155, 12.00000000].$$

The nonsupport zero moved from the point 0.6 to the point 0.5699639231. The optimal terminal state of the system was (0.20888225062, 3.439458969) and the objective function was 3.439458969.

3. TERMINAL OPTIMAL CONTROL PROBLEM

Return to problem (3), which is the main subject of this paper. Along with the terminal problem (3), consider the maximum excitation problem

$$\begin{gathered} c'(\nu)x(t^*) \longrightarrow \max, \\ \dot{x} = A(t)x + b(x)u, \quad x(t_*) = x_0, \quad \dot{u} = au + v, \quad u(t_*) = u_0, \\ |v(t)| \le M, \quad t \in T, \quad |u(s)| \le L, \quad s \in S_h, \end{gathered} \tag{24}$$

in which $c(\nu) = c - H'\nu$ and $\nu \in \mathbb{R}^m$ is the Lagrange vector. Since we have already described a technique for solving problems (24) in the class of piecewise constant control signals (see Subsection 2.5), we consider both problems in this class of feasible controls.

Let $u^0(t)$ $(t \in T)$ be the optimal open-loop control in problem (3), and $u^0(t\,|\,\nu)$ $(t \in T)$ be the optimal open-loop control in problem (24). By the Lagrange multiplier rule, there exists a vector $\nu^0 \in \mathbb{R}^m$ such that $u^0(t) \equiv u^0(t\,|\,\nu^0)$ $(t \in T)$.

In order to find the optimal Lagrange vector ν^0, we complement the system of refinement equations derived in Subsection 2.5 by the new variable ν and by the equation that follows from the terminal constraint:

$$Hx(t^*) - g = \int_{t_*}^{t^*} g(\vartheta)v(\vartheta)d\vartheta + G(t_*)x_0 + g(t_*)u_0 - g = 0. \tag{25}$$

Here, $G(t)$ and $g(t)$ $(t \in T)$ are the m-by-n matrix function and m-dimensional vector function obtained by solving the equations

$$\dot{G} = -GA, \quad \dot{g} = -ag - Gb(t)$$

respectively, with the initial conditions

$$G(t^*) = H, \quad g(t^*) = 0.$$

We write Eq. (25) in the form $R_0(y) = 0$, $y = (y^1, \dots, y^{l^*+1})$, where, according to (16),

$$R_0(y) = \sum_{l=1}^{l^*+1} \sum_{i=0}^{n^l} (-1)^i \gamma^l M \int_{t_i^l}^{t_{i+1}^l} g(\vartheta)d\vartheta - \sum_{l=1}^{s^l} au_l \int_{s_l}^{s^l} g(\vartheta)d\vartheta + G(t_*)x_0 + g(t_*)u_0 - g.$$

Therefore, the refinement equations are

$$R_0(y) = 0, \quad R_l(\nu, y^l) = 0, \quad l = \overline{1, l^*+1}. \tag{26}$$

It must be taken into account that the functions $\psi(t) = \psi_c(t) - \nu' G(t)$ $(t \in T)$ should be used instead of the functions $\psi_c(t)$ $(t \in T)$.

If matrices (21) are nonsingular, the Jacobian of system (26) is also nonsingular:

$$I(\nu, y) = \begin{pmatrix} 0_{m\times m} & A_1(y) \\ A_2(y) & \operatorname{diag}(I_l(\nu, y^l), l = \overline{1, l^*+1}) \end{pmatrix}.$$

Here, $A_1(y) = (A_1^l(y^l), l = \overline{1, l^*+1})$, $A_2(y) = (A_2^l(y), l = \overline{1, l^*+1})'$,

$$A_1^1(y^1) = (-2(-1)^i \gamma^1 Mg(t_i^1), i = \overline{1, n^1}, -g(s_1)(au_1 + (-1)^{n^1}\gamma^1 M)),$$

$$A_1^l(y^l) = (-g(s^{l-1})(au_{l-1} + \gamma^l M), -2(-1)^i \gamma^l Mg(t_i^1), i = \overline{1, n^l}, -g(s_l)(au_l + (-1)^{n^l}\gamma^l M)), \quad l = \overline{2, l^*},$$

$$A_1^{l^*+1}(y^{l^*+1}) = (-g(s^{l^*})(au_{l^*} + \gamma^{l^*+1}M), -2(-1)^i \gamma^{l^*+1} Mg(t_i^{l^*+1}), i = \overline{1, n^{l^*+1}}),$$

$$A_2^1(y^1) = \left(\int_{t_i^1}^{s_1} G(\vartheta)b(\vartheta)\Phi(\vartheta - t_i^1)d\vartheta,\ i = \overline{1, n^1},\ 0\right)',$$

$$A_2^l(y^l) = \left(\int_{s^{l-1}}^{s_l} G(\vartheta)b(\vartheta)\Phi(\vartheta - s^{l-1})d\vartheta, \int_{t_i^l}^{s_l} G(\vartheta)b(\vartheta)\Phi(\vartheta - t_i^1)d\vartheta,\ i = \overline{1, n^l},\ 0_{m\times 1}\right)', \quad i = \overline{2, l^*},$$

$$A_2^{l^*+1}(y^{l^*+1}) = \left(\int_{s^{l^*}}^{t^*} G(\vartheta)b(\vartheta)\Phi(\vartheta - s^{l^*})d\vartheta, \int_{t_i^{l^*+1}}^{t^*} G(\vartheta)b(\vartheta)\Phi(\vartheta - t_i^{l^*+1})d\vartheta,\ i = \overline{1, n^{l^*+1}}\right)'.$$

In order to quickly find the elements required for solving the refinement equations by the Newton method, we calculate and store in memory (in addition to integrals (22)) the integral

$$\int_{t_i^{0l}(h)}^{s_l^0(h)} G(\vartheta)b(\vartheta)\Phi(\vartheta - s^{0l-1}(h))d\vartheta, \quad i = \overline{0, n^{0l}}, \quad l = \overline{1, l^*+1},$$

and the values $G(t)$ and $g(t)$ for $t \in T_0^l$, $l = \overline{1, l^*+1}$, $p = Hx^0(t^*) - g$.

It is well known that the Newton method converges if the initial approximations are sufficiently good. If a good approximation of the Lagrange vector ν is given, the initial approximations of y are found by solving problem (24) in the class of discrete (or piecewise continuous) control signals. The initial approximation $\nu^{(1)}$ is found by solving the terminal problem (3) driven directly by the bounded control $|u(t)| \le L$ ($t \in T$). Then, this approximation is iteratively improved by the formula

$$\nu^{(k+1)} = \nu^{(k)} + \theta_k(Hx(t^* \mid \nu^{(k)}) - g),$$

where θ_k is the step found by any gradient method (for example, the coordinatewise descent method) and $x(t^* \mid \nu)$ is the optimal terminal state in problem (24) for $\nu = \nu^{(k)}$. Upon constructing a satisfactory initial approximation, we solve the refinement equations by the Newton method (26).

4. SYNTHESIS OF FEEDBACK OPTIMAL CONTROLS

Define the concept of the feedback solution to problem (3). Assume that the exact state of system (1) and the values of the control are not only available at the initial time $t = t_*$, but will also be available at every instance $\tau \in T_{h_0}$ ($h_0 = (t^* - t_*)/N_0$, $N_0 > 0$) in the course of the control process. Under this assumption, we embed problem (3) in the family of problems

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(\tau) = z, \quad \dot{u} = au + v, \quad u(\tau) = y, \\ Hx(t^*) = g, \quad |v(t)| \le M, \quad t \in T(\tau) = [\tau, t^*], \quad |u(s)| \le L, \quad s \in T(\tau), \end{gathered} \tag{27}$$

which depend on the scalars $\tau \in T_{h_0}$, y and the n-dimensional vector z. The triple $(\tau; z, y)$ is called the position of the system.

Let $v^0(t \mid \tau, z, y)$ ($t \in T(\tau)$) be the optimal open-loop control signal in problem (27) for the position $(\tau; z, y)$ and G_τ be the set of all pairs (z, y) for which problem (27) has a solution.

The function

$$v^0(\tau, z, y) = (v^0(t \mid \tau, z, y), t \in [\tau, \tau + h_0[), \quad (z, y) \in G_\tau, \quad \tau \in T_{h_0},$$

is called the optimal feedback control signal, and the construction of this signal is termed the synthesis of the optimal feedback (synthesis of the optimal system).

For nontrivial cases, no analytic solution of the synthesis problem is possible. The classical maximum principle and dynamic programming cannot cope with this problem either. Therefore, following [9, 10], we describe a method for constructing a realization of the optimal feedback in the course of a particular control process.

Assume that the optimal feedback $v^0(t, x, u)$ ($(x, u) \in G_t$, $t \in T_{h_0}$) is already constructed. Since this feedback is used to control a real-life system (although it was constructed on the basis of an "ideal" mathematical model) and was designed to counteract the inaccuracies of the model and unknown perturbations, we consider the feedback system in the form

$$\begin{aligned} \dot{x} &= A(t)x + b(t)u + w, \quad x(t_*) = x_0, \\ \dot{u} &= au + v^0(t, x, u), \quad u(t_*) = u_0. \end{aligned} \tag{28}$$

Here, $w = w(t, x, u)$ ($(x, u) \in G_t$, $t \in T$) is the set of terms that represent the unknown perturbation and inaccuracies of the model; along every particular $x = x(t)$, $u = u(t)$ ($t \in T$), this set of terms is realized as a piecewise continuous n-dimensional function $w(t) = w(t, x(t), u(t))$ ($t \in T$).

By the trajectory of the nonlinear system (28), we mean the solution to the linear system

$$\dot{x} = A(t)x + b(t)u + w, \quad x(t_*) = x_0, \quad \dot{u} = au + v^0(t), \quad u(t_*) = u_0,$$

driven by the control signal $v^0(t) = v^0(t \mid \tau, x(\tau), u(\tau))$ ($t \in [\tau, \tau + h[$, $\tau \in T_{h_0}$). Under this approach to the definition of the solution to Eq. (28), there is no question of the existence of a solution to the differential equation with a discontinuous (in state) right-hand side.

Consider a particular control process. Assume that a piecewise continuous perturbation $w^*(t)$ ($t \in T$), which is not known in advance, is realized in the course of this process. It induces a particular trajectory $x^*(t)$ ($t \in T$) and a particular control $u^*(t)$ ($t \in T$) of the closed system (28) that satisfy the identities

$$\dot{x}^* \equiv A(t)x^*(t) + b(t)u^*(t) + w^*(t), \quad x^*(t_*) = x_0, \quad \dot{u}^* \equiv au^* + v^0(t, x^*(t), u^*(t)), \quad u^*(t_*) = u_0.$$

It is seen from these identities that the entire (for all $(x, u) \in G_t$, $t \in T_{h_0}$) optimal feedback is not used for control; rather, only its values $v^*(t) = v^0(t \mid \tau, x^*(\tau), u^*(\tau))$ ($t \in [\tau, \tau + h[$, $\tau \in T_{h_0}$) along the continuous trajectory $x^*(t)$ ($t \in T$) and continuous control $u^*(t)$ ($t \in T$) are used. The function $v^*(t)$ ($t \in T$) is called the realization of the optimal feedback in the particular control process. If the time needed to calculate $v^*(\tau)$ at every instance $\tau \in T_{h_0}$ does not exceed h_0, we say that the realization of the optimal feedback is constructed in real time. A device that performs such a calculation in real time is called the optimal controller. Operation in real time means that the rate of producing control signals is the same as the measurement rate. However, the instance of time at which the control signal is produced can differ from the instance at which the corresponding measurement arrives. In this paper, we assume that the performance of the hardware used to process the measurements is sufficiently high to neglect this time lag. The realization of optimal feedbacks in real time in the case of nonzero time lag, based on the use of several processors, will be considered in another paper. When performance of the hardware is not sufficiently high, one can use mixed open-loop–feedback solutions based on subsampled measurements.

Thus, the problem of synthesizing the optimal feedback control signal is reduced to the construction of an algorithm for the optimal controller.

We derive this algorithm from the definition of the optimal feedback and the refinement equations (see Section 3). Assume that the desired algorithm is already constructed, the controller has operated on the interval $[t_*, \tau[$ and produced the control signals $v^*(t)$ ($t \in [t_*, \tau[$) that, in turn, induced the control $u^*(t)$ ($t \in [t_*, \tau[$). Assume that the control system driven by this control and the perturbation $w^*(t)$ ($t \in [t_*, \tau]$) arrived at the state $x^*(\tau)$ at the current time τ. In order to find $v^*(t)$ ($t \in [\tau, \tau + h_0[$), the optimal controller must quickly find the open-loop solution to the problem

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \\ \dot{x} = A(t)x + b(t)u, \quad x(\tau) = x^*(\tau), \quad \dot{u} = au + v, \quad u(\tau) = u^*(\tau), \\ Hx(t^*) = g, \quad |v(t)| \le M, \quad t \in T(\tau) = [\tau, t^*], \quad |u(s)| \le L, \quad s \in T(\tau). \end{gathered} \tag{29}$$

It is known that the optimal controller has already solved a problem similar to (29) at the previous interval $[\tau - h_0, \tau[$; in this problem the process started at $\tau - h_0$ from the state $x^*(\tau - h_0)$ with the control $u^*(\tau - h_0)$. The controller solved this problem (found the optimal structure) $(\nu(\tau - h_0), y(\tau - h_0))$.

Denote by $y_1(\tau - h_0)$ the first component of the vector $y(\tau - h_0)$; i.e., $y_1(\tau - h_0) = t_1^1(\tau - h_0)$ or $y_1(\tau - h_0) = s_1(\tau - h_0)$, or $y_1(\tau - h_0) = s^1(\tau - h_0)$. We say that the structure of the control signal $v^0(t \mid \tau - h_0, x^*(\tau - h_0), u^*(\tau - h_0))$ $(t \in T(\tau - h_0))$ is not violated at the time τ if

$$\tau < y_1(\tau - h_0). \tag{30}$$

Under condition (30), we take $(\nu(\tau - h_0), y(\tau - h_0))$ as the initial approximation for solving the refinement equations by the Newton method: $(\nu^{(1)}(\tau), y^{(1)}(\tau)) = (\nu(\tau - h_0), y(\tau - h_0))$. We also take into account that

$$R_l(\nu^{(1)}(\tau), y^{(1)}(\tau)) = R_l(\nu(\tau - h_0), y(\tau - h_0)) = 0, \quad l = \overline{1, l^*(\tau - h_0) + 1},$$

$$R_0(y^{(1)}(\tau)) = R_0(y(\tau - h_0)) + G(\tau)x^*(\tau) + g(\tau)u^*(\tau) - G(\tau - h_0)x^*(\tau - h_0) - g(\tau - h_0)u^*(\tau - h_0)$$
$$- v^*(\tau - h_0)\int\limits_{\tau - h_0}^{\tau} g(\vartheta)d\vartheta,$$

and the Jacobians $I(\nu^{(1)}(\tau), y^{(1)}(\tau))$ and $I(\nu(\tau - h_0), y(\tau - h_0))$ are identical except for the element A_{11}^1, which is now $-\Phi(s_1 - \tau)\,(au^*(\tau) + \gamma^1(\tau - h_0)M)$.

The desired values are easily obtained by a single integration over the interval $[\tau - h_0, \tau]$ provided that $G(\tau - h_0)$, $g(\tau - h_0)$, $x^*(\tau - h_0)$, and $u^*(\tau - h_0)$ are stored in computer memory in addition to other data.

If condition (30) does not hold, we set $\nu^{(1)}(\tau) = (\nu(\tau - h_0), y^{(1)}(\tau)) = (y_2(\tau - h_0), \dots, y_{i^*}(\tau - h_0))$ (where i^* is the number of components of the vector $y(\tau - h_0)$); $\gamma^1(\tau - h_0) = -\gamma^1(\tau - h_0)$ if $y_1(\tau - h_0) = t_1^1(\tau - h_0)$; $\gamma^1(\tau - h_0)$ is undefined if $y_1(\tau - h_0) = s_1(\tau - h_0)$, and $\gamma^1(\tau - h_0) = \gamma^2(\tau - h_0)$ if $y_1(\tau - h_0) = s_1(\tau - h_0)$. $R_0(y^{(1)}(\tau))$, $R_l(\nu^{(1)}(\tau))$, $y^{(1)})$ $(l = \overline{1, l^*(\tau) + 1})$, and the matrix $I(\nu^{(1)}(\tau), y(1))$ are constructed as in the preceding case with the only difference being that the $(m + 1)$st equation must be removed from the system of equations and the $(m + 1)$st row and column must be removed from the Jacobi matrix.

Since problems (29) set up for the instances $\tau - h_0$ and τ are close to each other, the initial approximations $(\nu^{(1)}(\tau), y^{(1)}(\tau))$ are close to the exact solution $(\nu(\tau), y(\tau))$ of the refinement equations if h_0 is small. Usually, two or three iterations of the Newton method are sufficient to obtain a very accurate solution.

Given the switching points $y(\tau)$ and the numbers $\gamma^l(\tau - h_0)$, $u_l(\tau - h_0)$ $(l \in l^*(\tau - h_0))$, we construct the structure of the optimal open-loop control signal $v^0(t \mid \tau, x^*(\tau), u^*(\tau))$ $(t \in T(\tau))$ by the following rules:

$$\tau < y_1(\tau) : l^*(\tau) = l^*(\tau - h_0), \quad \gamma^l(\tau) = \gamma^l(\tau - h_0), \quad l = \overline{1, l^*(\tau)},$$
$$u_l(\tau) = u_l(\tau - h_0), \quad l = \overline{1, l^*(\tau)};$$
$$\tau \ge y_1(\tau) = t_1^1(\tau) : l^*(\tau) = l^*(\tau - h_0), \quad t_i^1(\tau) = t_{i+1}^1(\tau), \quad i = \overline{1, n^1(\tau - h_0) - 1},$$
$$\gamma^1(\tau) = -\gamma^1(\tau - h_0), \quad \gamma^l(\tau) = \gamma^l(\tau - h_0), \quad l = \overline{2, l^*(\tau)},$$
$$u_l(\tau) = u_l(\tau - h_0), \quad l = \overline{1, l^*(\tau)};$$
$$\tau \ge y_1(\tau) = s_1(\tau) : l^*(\tau) = l^*(\tau - h_0), \quad s_1(\tau) = \tau, \quad \gamma^1(\tau) \text{ is undefined},$$
$$\gamma^l(\tau) = \gamma^l(\tau - h_0), \quad l = \overline{2, l^*(\tau)}, \quad u_l(\tau) = u_l(\tau - h_0), \quad l = \overline{1, l^*(\tau)};$$
$$\tau \ge y_1(\tau) = s^1(\tau) : l^*(\tau) = l^*(\tau - h_0) - 1, \quad S^l(\tau) = S^{l+1}(\tau - h_0), \quad l = \overline{1, l^*(\tau)},$$
$$\gamma^l(\tau) = \gamma^{l+1}(\tau - h_0), \quad l = \overline{1, l^*(\tau)}; \quad u_l(\tau) = u_{l+1}(\tau - h_0), \quad l = \overline{1, l^*(\tau)}.$$

Using rules (16), we reconstruct the optimal open-loop control signal from the structure; this signal $v^0(t \mid \tau, x^*(\tau), u^*(\tau))$ $(t \in [\tau, \tau + h_0[)$ will be used on the interval $[\tau, \tau + h_0[$ as the realization of the optimal feedback.

It was shown in Section 3 that Newton iterations could be performed quickly due to storing specially selected data in computer memory. The complexity of the computation of $v^*(\tau)$ at the time τ cannot be expressed explicitly in terms of the parameters of the system (Section 4). For a particular case, it can be calculated by the formula

$$E(\tau) = \left(h_0 + \sum_{k=1}^{k^0(\tau)} \max_{i=1,i^*} \left|y_i^{(k)}(\tau) - y_i^{(k-1)}(\tau)\right|\right)(t^* - t_*)^{-1},$$

where $y_i^{(k)}(\tau)$ is the ith component of the vector $y(\tau)$ at the kth iteration of the Newton method and $k^0(\tau)$ is the number of iterations. One can gain an impression of the value of $E(\tau)$ from the nontrivial example discussed below.

Remark 5. The algorithm for constructing realizations of the optimal feedback is designed so as to take advantage of specific features of the problem and decrease the complexity of its solution. Therefore, we claim that modern processors used to implement the optimal controller can cope with the task in a short amount of time $s(\tau)$ ($\tau \in T_{h_0}$) such that its influence on the trajectory $x^*(t)$ ($t \in T$) is negligible. Since there are no perturbations in the actuator, we can assume that the actual control $u^*(t)$ ($t \in T$) coincides with $u^0(t)$ ($t \in T$), where $u^0(t) = u^0(t \mid \tau, x^*(\tau), u^*(\tau))$ ($t \in [\tau, \tau + h_0[$, $\tau \in T_{h_0}$); here, $u^0(t \mid \tau, x^*(\tau), u^*(\tau))$ ($t \in T(\tau)$) is the optimal open-loop control in problem (29). Therefore, the constraints $|u^*(t)| \le L$ ($t \in T$) are satisfied.

If the delay time is substantial, then $u^*(t) \not\equiv u^0(t)$ ($t \in T$), which can result in the violation of phase constraints. This problem will be studied in another paper.

Example 2. Consider problem (10) formulated in Subsection 2.4 supplemented with the additional terminal constraint:

$$\begin{gathered} x_2(12) \longrightarrow \max, \\ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad x_1(0) = x_2(0) = 0, \quad x_1(12) = 2, \\ \dot{u} = -u + v, \quad u(0) = 0, \\ |u(t)| \le 0.05, \quad |v(t)| \le 1, \quad t \in T = [0, 12]. \end{gathered} \tag{31}$$

When solving the refinement equations, we used the initial approximation $\nu = 0$, and the initial approximation of the vector y was set to the switching points of the control signal obtained by solving problem (10).

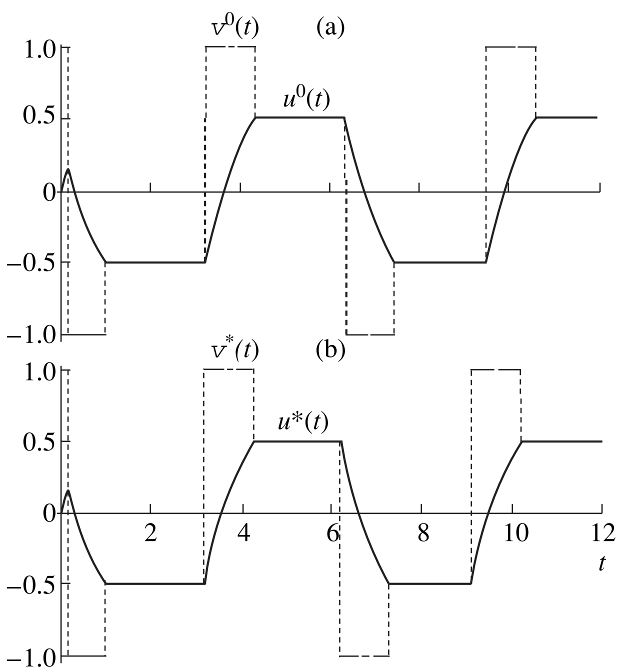

Fig. 4.

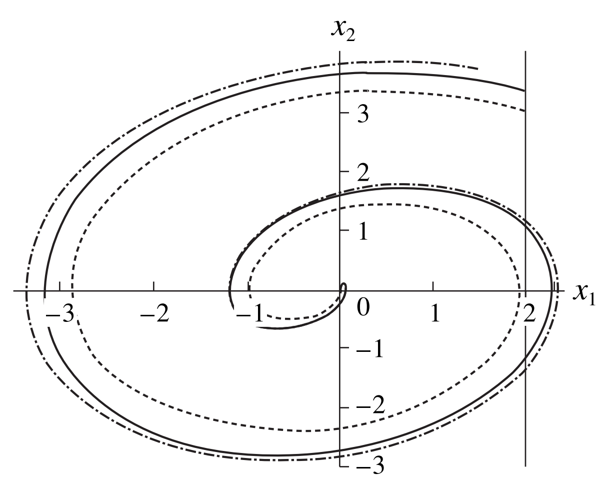

Fig. 5.

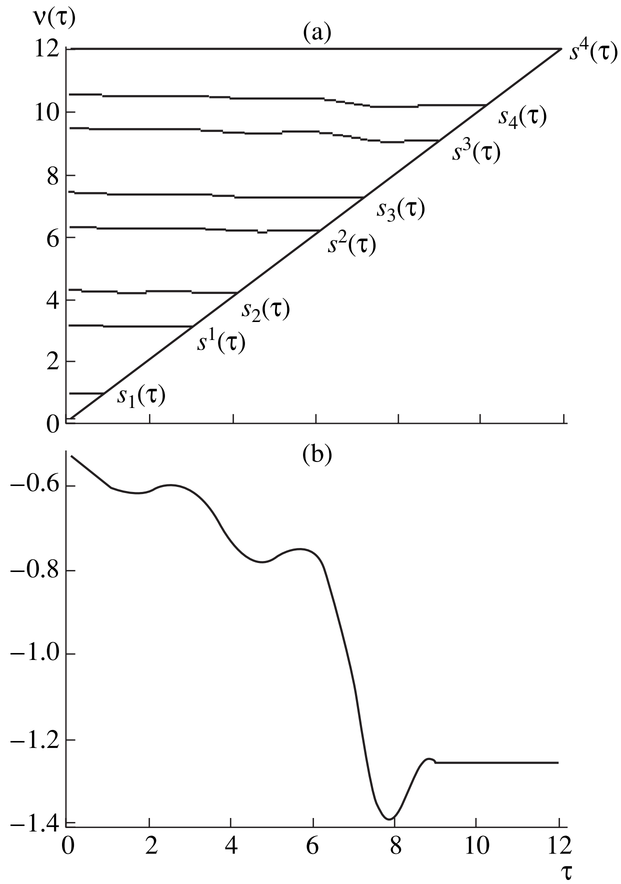

Fig. 6.

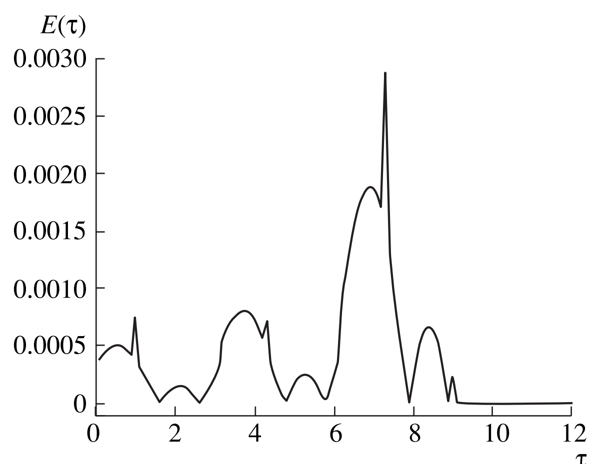

Fig. 7.

In five iterations of the Newton method, we reached the residual of the terminal constraint of the order of magnitude $\varepsilon = 10^{-9}$ and obtained the following values:

$$\nu = -0.5236204081, \quad t_1^1 = 0.1632167993,$$

$$s_1 = 0.9966414272, \quad s^1 = 3.214849066, \quad s_2 = 4.313461355, \quad s^2 = 6.356441719,$$

$$s_3 = 7.455054008, \quad s^3 = 9.498034373, \quad s_4 = 10.59664666, \quad s^4 = 12.0.$$

Figure 4a shows the optimal control signal $v^0(t)$ $(t \in T)$, the control $u^0(t)$ $(t \in T)$. The phase trajectory on the plane x_1x_2 is shown in Fig. 5 (the dotted curve). At the terminal instance of time, the system arrived at the state $x_1(12) = 2.0$, $x_2(12) = 3.003578347$; i.e., the objective function was 3.003578347.

Let us construct a realization of the optimal feedback for a particular control process in problem (31). Assume that the dynamic system is affected by a piecewise continuous perturbation and the system is described by the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u + w.$$

Assume that the perturbation that is realized in the control process is

$$w^*(t) = -0.2\cos(t), \quad t \in [0, 9], \quad w^*(t) \equiv 0, \quad t \in]9, 12].$$

This perturbation is not known to the controller; however, the current state of the system $x^*(t)$ and the control $u^*(t)$ at every instance of time $t \in T_h$ are available to the actuator.

The realization of the optimal feedback $v^*(t)$ $(t \in T)$ in the process under consideration is illustrated in Fig. 4b. Figure 5 shows the phase trajectories of system (10) on the plane x_1x_2 induced by the optimal open-loop control signal $v^0(t)$ $(t \in T)$ in the absence of perturbations (dotted curve), by the control signal $v^0(t)$ $(t \in T)$ perturbed by $w^*(t)$ $(t \in T)$ (dot-and-dash curve), and by the optimal feedback subject to the perturbation $w^*(t)$ $(t \in T)$ (solid curve). It is clear that, in the presence of perturbations, the optimal open-loop control signal cannot even bring system (31) to the terminal state. The realization of the optimal feedback brings system (31) to the state $x_1(12) = 2.0$, $x_2(12) = 3.350068609$. The motion of the elements of the sets $S_s(\tau)$, $T_0^l(\tau)$ $(l = \overline{1, l^*}, \tau \in T_h)$ is shown in Fig. 6a, and the change of $\nu(\tau)$ $(\tau \in T_h)$ is shown in Fig. 6b.

Figure 7 shows the complexity $E(\tau)$ of computing $v^*(t)$ $(t \in [\tau, \tau + h[, \tau \in T_h)$. It is seen that the computational complexity does not exceed 0.003. Modern processors can do this job in less than $h = 0.1$ (if the real time in problem (10) is measured in, e.g., seconds).

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REFERENCES

1. L. S. Pontryagin, V. G. Boltyanskiĭ, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes* (Nauka, Moscow, 1983; Interscience, New York, 1962).
2. A. A. Fel'dbaum, *Foundations of the Theory of Optimal Automatic Systems* (Fizmatgiz, Moscow, 1963) [in Russian].
3. A. Ya. Dubovitskiĭ and A. A. Milyutin, Zh. Vychisl. Mat. Mat. Fiz. **5**, 395 (1965).
4. N. V. Balashevich, R. Gabasov, and F. M. Kirillova, Zh. Vychisl. Mat. Mat. Fiz. **40**, 838 (2000) [Comput. Math. Math. Phys. **40**, 799 (2000)].
5. R. Gabasov, N. M. Dmitruk, and F. M. Kirillova, Avtom. Telemekh., No. 3, 3 (2002) [Autom. Remote Contr., No. 3, 345 (2002)].
6. R. Gabasov, F. M. Kirillova, and A. I. Tyatyushkin, *Constructive Optimization Methods* (Universitetskoe, Minsk, 1984) Part 1 [in Russian].
7. R. Gabasov and F. M. Kirillova, *Optimization of Linear Systems* (Belarussian State University, Minsk, 1973) [in Russian].
8. G. B. Dantzig, *Linear Programming and Extensions* (Princeton University Press, Princeton, 1963; Progress, Moscow, 1966).
9. R. Gabasov, F. M. Kirillova, and O. I. Kostyukova, Dokl. Akad. Nauk SSSR **320**, 1294 (1991).
10. R. Gabasov, F. M. Kirillova, and S. V. Prishchepova, in Lecture Notes in Control and Information Sciences, Ed. by M. Thoma, (Springer, Berlin, 1995) vol. 207.